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A MARKOV PROCESS FOR SEQUENTIAL ALLOCATION

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Abstract: We describe a Markov process which models the sequential allocation for two adjacent tables coexisting in memory by growing towards each other. The tables are expected to fill at the same rate; random deletions and insertions are allowed.

1 Introduction

A widespread technique for representing two variable-size sequential lists in memory is to store them in reverse order, so that list 1 expands to the right and list 2 expands to the left. Overflow occurs only when the total size of both lists exhausts the available space.

D. E. Knuth proposed in [4] the following mathematical model: fluctuations in the tables are represented by a finite sequence of insertion and deletion operations $a_1, a_2, \ldots, a_n$, where each $a_i \in \{1, 2\}$ is interpreted with probability p as a deletion and with probability $1-p$ as an insertion on top of stack a_i. A deletion from an empty list has no effect. Let l_1 and l_2 be the respective sizes of the tables during this process, k_1 and k_2 their sizes after the memory is full and m the number of memory locations available. The process continues until $l_1 + l_2 = m$. The purpose of this paper is to examine the dependence of $\max(k_1, k_2)$ on p and m and to determine formulæ for the probability distribution of (k_1, k_2).

2 A Mathematical Model

We introduce a finite state Markov process modelling the random fluctuations in the adjacent tables. The set of states is

$$S = \{\{a, b\} \mid 0 \le a \le b \le m, 0 \le a + b \le m\}.$$

The system has n_m states, where

$$n_m = \sum_{k=0}^{m} ([k/2] + 1) = \left(\left[\frac{m}{2}\right] + 1\right)\left(\left[\frac{m+1}{2}\right] + 1\right).$$

We label the states of the system in a manner similar to Cantor's diagonalization method; state $\{a, b\}$ comes before state $\{c, d\}$ if $a + b < c + d$ or $a + b = c + d$ and $a < c$. The system is in state $\{a, b\}$ if $l_1 = a$, $l_2 = b$ or $l_1 = b$, $l_2 = a$.

We denote by $\{a, b\} \to \{c, d\}$ a transition from state $\{a, b\}$ to state $\{c, d\}$ and define the transition probabilities according to the following deletion/insertion rules :

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$P(\{0,0\} \to \{0,0\}) = p$ and $P(\{0,0\} \to \{0,1\}) = 1-p$

$P(\{a,b\} \to \{c,d\}) = 0$ if $c \notin \{a-1,a,a+1\}$ or $d \notin \{b-1,b,b+1\}$.

$P(\{a,b\} \to \{a,b\}) = 1 \quad$ if $a+b=m$

$P(\{a,b\} \to \{a,b\}) = p/2 \quad$ if $a=0,\ b\neq 0,\ b\neq m$

$P(\{a,b\} \to \{a-1,b\}) = p/2 \quad$ if $a\neq 0,\ a\neq b,\ a+b\neq m$

$P(\{a,b\} \to \{a-1,b\}) = p$ and $P(\{a,b\} \to \{a,b+1\}) = 1-p \quad$ if $a=b\neq 0$

$P(\{a,b\} \to \{a,b-1\}) = p/2$ and $P(\{a,b\} \to \{a+1,b\}) = P(\{a,b\} \to \{a,b+1\}) = \frac{1-p}{2}$ if $a\neq b,\ a+b\neq m$

Let $k_m = [m/2]+1$. The process begins in the state $\{0,0\}$ when both lists are empty and continues until the memory is full, that is until one of the last k_m states with $l_1+l_2=m$ is reached. Therefore we may regard each of these states as an absorbing state; the states $1,2,\ldots,n_m-k_m$ are transient because they certainly will not be occupied when the process stops (their limiting probability is null).

We are interested in the limiting state probability of absorbing states which represent the probability distribution of (k_1,k_2). They can be obtained from the theory of canonical decomposition of matrices using a procedure of approximation of characteristic roots outlined in [1]. Since our transition matrix is large and sparse we shall instead use the technique of the generating function described in [2] to determine the asymptotic behavior of transition probabilities.

Let $\pi(n)$ be the row vector with components $\pi_i(n)$, where $\pi_i(n)$ is the probability that the system will occupy state i after n transitions. Then

$$\pi_j(n+1) = \sum_{i=1}^{n_m} \pi_i(n)p_{ij}, \quad n=0,1,2,\ldots$$

and

$$\pi(n+1) = \pi(n)P, \quad n\in \mathbb{N} \tag{1}$$

where $P=(p_{i,j})_{1\le i,j\le n_m}$ is the transition probability matrix.

Let $\Pi(z)$ denote the z-transform of the vector $\pi(n)$. From (1) we obtain

$$z^{-1}[\Pi(z)-\pi(0)] = \Pi(z)P$$

and through rearrangement

$$\Pi(z) = \pi(0)(I-zP)^{-1} \tag{2}$$

where I is the identity matrix of order n_m. Let the matrix $H(n)$ be the inverse transform of $(I-zP)^{-1}$ and $H(n)[i,j]$ the element at the intersection of row i and column j in $H(n)$. Taking the inverse transform of (2) we obtain

$$\pi(n) = \pi(0)H(n)$$

Since the process begins in the first state the initial state probabilities are $\pi(0) = (1,0,\ldots,0)$, so that

$$\pi_i(n) = H(n)[1,i], \quad 1\le i\le n_m. \tag{3}$$

R. Howard shows in [3] that $H(n) = S + T(n)$, where the matrix $T(n)$ representing the transient behavior of the process tends to zero as n becomes very large. The stochastic matrix S is the steady-state component that arises from a term of $(I - zP)^{-1}$ of the form $1/(1-z)$. The i-th row of S represents the limiting state probability distribution that would exist if the system were started in the i-th state.

The last k_m rows of matrix P are the vectors $e_{n_m - k_m + 1}, \ldots, e_{n_m}$ from the canonical base of $\mathbb{R}^{n_m}$, therefore we can express P as

$$P = \begin{pmatrix} A & B \\ 0 & I \end{pmatrix}$$

where the region 0 consists entirely of zeroes, I is the identity matrix of k_m dimension, $A \in \mathcal{M}_{n_m - k_m}(\mathbb{R}[p])$, $B \in \mathcal{M}_{n_m - k_m, k_m}(\mathbb{R}[p])$. (Here $\mathbb{R}[p]$ denotes the ring of real polynomials in the variable p.)
Hence we obtain

$$I - zP = \begin{pmatrix} I - zA & -zB \\ 0 & (1-z)I \end{pmatrix}.$$

¿From Laplace's rule we have

$$\det(I - zP) = \det(I - zA) \cdot \det((1-z)I) = (1-z)^{k_m} \widetilde{Q}(z,p)$$

where we have denoted $\det(I - zA) = \widetilde{Q}(z,p) \in \mathbb{R}[z,p]$, $\deg_p \widetilde{Q}(z,p) = n_m - k_m$.

Lemma 1. *The relation $\widetilde{Q}(1,p) \neq 0$ holds true.*

Proof. In any finite Markov chain the probability that after n steps the process is in an ergodic state tends to 1 as n tends to infinity. The $(n_m - k_m) \times (n_m - k_m)$ matrix A concerns the process as long as it stays in transient states, hence its powers tend to 0. We consider the identity

$$(I - A) \cdot (I + A + A^2 + \cdots + A^{n-1}) = I - A^n,$$

whose right side tends to I. Thus, for sufficiently large n, the determinant of the matrix $I - A^n$ must be non-zero, therefore the determinant of the matrix $I - A$ cannot be zero.
But $\widetilde{Q}(1,p) = \det(I - A)$, so that $\widetilde{Q}(1,p) \neq 0$, which completes the proof.

Let $R_i = \lim_{n\to\infty} \pi_i(n)$ be the probability of an absorption in the state i when the system has started from state 1. From (3) it follows that $R_i = \lim_{n\to\infty} H(n)[1,i]$. Now we are able to prove the following

Theorem 2.1 *There exist $Q(p) \in \mathbb{R}[p]$ and $Q_i(p) \in \mathbb{R}[p]$, $n_m - k_m \leq i \leq n_m$, with* $\deg Q = n_m - k_m$, $\deg Q_i < n_m - k_m$ *such that*

$$R_i = \frac{Q_i(p)}{Q(p)}, \quad i = n_m - k_m + 1, \ldots, n_m.$$

Proof. Let M_i be the matrix obtained from matrix $I - zP$ by deleting the first column and the i-th row. Therefore $\det(M_i) = (1-z)^{k_m-1}\widetilde{Q}_i(z,p)$ for $n_m - k_m \leq i \leq n_m$, where $\widetilde{Q}_i(z,p) \in I\!R[z,p]$, $\deg_p \widetilde{Q}_i(z,p) < n_m - k_m$. We have

$$(I-zP)^{-1}[1,i] = \frac{(-1)^{i+1}\det M_i}{\det(I-zP)} = \frac{(-1)^{i+1}(1-z)^{k_m-1}\widetilde{Q}_i(z,p)}{(1-z)^{k_m}\widetilde{Q}(z,p)} = \frac{(-1)^{i+1}\widetilde{Q}_i(z,p)}{(1-z)\widetilde{Q}(z,p)}$$

which is a function of z with a factorable denominator. By partial fraction expansion

$$\frac{(-1)^{1+i}\widetilde{Q}_i(z,p)}{(1-z)\widetilde{Q}(z,p)} = \frac{T_i(p)}{1-z} + \frac{S_i(z,p)}{\widetilde{Q}(z,p)}$$

where $T_i(p)$ is a rational fraction in the variable p. It follows that

$$T_i(p)\widetilde{Q}(z,p) + (1-z)S_i(z,p) = (-1)^{i+1}\widetilde{Q}_i(z,p).$$

For $z = 1$ we obtain $T_i(p)\widetilde{Q}(1,p) = (-1)^{i+1}\widetilde{Q}_i(1,p)$. Since by lemma (1) $\widetilde{Q}(1,p) \neq 0$, we have

$$T_i(p) = (-1)^{i+1}\frac{\widetilde{Q}_i(1,p)}{\widetilde{Q}(1,p)}, \quad n_m - k_m < i \leq n_m. \tag{4}$$

Let us make the notation $Q_i(p) = (-1)^{i+1}\widetilde{Q}_i(1,p)$ and $Q(p) = \widetilde{Q}(1,p)$. Since $H(n)$ is the inverse transform of $(I-zP)^{-1}$ and $R_i = \lim_{n\to\infty} H(n)[1,i]$, we have from [3] that R_i equals the coefficient of $\frac{1}{1-z}$ in the expansion of $(I-zP)^{-1}[1,i]$ and therefore $R_i = T_i(p)$. Now by (4) we have the desired representation of R_i.

In an absorbing chain the limiting probabilities of the transient states are zero, therefore the sum of the absorption probabilities is 1. Since the states $\{l_1, l_2\}$ with $l_1 + l_2$ are absorbing, it follows that $\sum_{i=n_m-k_m+1}^{n_m} R_i = 1$. But $R_i, n_m - k_m < i \leq n_m$ are the limiting probabilities of the states $\{k_1, k_2\}$, hence we have obtained a representation of the distribution of (k_1, k_2) depending on p.

3 Examples

In [4] Knuth mentions the probability distribution of (k_1, k_2) for $m = 4$ and notes that the difference between k_1 and k_2 tends to increase as p increases. We shall further analyze the case $m = 5$. The ordered set of states is $S = \{\{0,0\},\{0,1\},\{0,2\},\{1,1\},\{0,3\},\{1,2\},\{0,4\},\{1,3\},\{2,2\},\{0,5\},\{1,4\},\{2,3\}\}$. The transition probability matrix is

$$
P = \begin{pmatrix}
p & 1-p & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{p}{2} & \frac{p}{2} & \frac{1-p}{2} & \frac{1-p}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{p}{2} & \frac{p}{2} & 0 & \frac{1-p}{2} & \frac{1-p}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & p & 0 & 0 & 0 & 1-p & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{p}{2} & 0 & \frac{p}{2} & 0 & \frac{1-p}{2} & \frac{1-p}{2} & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{p}{2} & \frac{p}{2} & 0 & 0 & 0 & \frac{1-p}{2} & \frac{1-p}{2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{p}{2} & 0 & \frac{p}{2} & 0 & 0 & \frac{1-p}{2} & \frac{1-p}{2} & 0 \\
0 & 0 & 0 & 0 & \frac{p}{2} & \frac{p}{2} & 0 & 0 & 0 & 0 & \frac{1-p}{2} & \frac{1-p}{2} \\
0 & 0 & 0 & 0 & 0 & p & 0 & 0 & 0 & 0 & 0 & 1-p \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}.
$$

The quantities R_{10}, R_{11}, R_{12} are the limiting probability distribution of states $\{0,5\}, \{1,4\}, \{2,3\}$. We have

$$
\widetilde{Q}(1,p) = -\frac{(p^4 - 17p^3 + 68p^2 - 96p + 64)(p-1)^5}{64},
$$

$$
\widetilde{Q}_{10}(1,p) = \begin{vmatrix}
p-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1-\frac{p}{2} & \frac{p-1}{2} & \frac{p-1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{-p}{2} & 1-\frac{p}{2} & 0 & \frac{p-1}{2} & \frac{p-1}{2} & 0 & 0 & 0 & 0 \\
-p & 0 & 1 & 0 & p-1 & 0 & 0 & 0 & 0 \\
0 & \frac{-p}{2} & 0 & 1-\frac{p}{2} & 0 & \frac{p-1}{2} & \frac{p-1}{2} & 0 & 0 \\
0 & \frac{-p}{2} & \frac{-p}{2} & 0 & 1 & 0 & \frac{p-1}{2} & \frac{p-1}{2} & 0 \\
0 & 0 & 0 & \frac{-p}{2} & 0 & 1-\frac{p}{2} & 0 & 0 & \frac{p-1}{2} \\
0 & 0 & 0 & \frac{-p}{2} & \frac{-p}{2} & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & -p & 0 & 0 & 1 & 0
\end{vmatrix}
$$

$$
= \frac{(p^3 - p^2 + 4)(p-1)^5}{64},
$$

$$
\widetilde{Q}_{11}(1,p) = \begin{vmatrix}
p-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1-\frac{p}{2} & \frac{p-1}{2} & \frac{p-1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{-p}{2} & 1-\frac{p}{2} & 0 & \frac{p-1}{2} & \frac{p-1}{2} & 0 & 0 & 0 & 0 \\
-p & 0 & 1 & 0 & p-1 & 0 & 0 & 0 & 0 \\
0 & \frac{-p}{2} & 0 & 1-\frac{p}{2} & 0 & \frac{p-1}{2} & \frac{p-1}{2} & 0 & 0 \\
0 & \frac{-p}{2} & \frac{-p}{2} & 0 & 1 & 0 & \frac{p-1}{2} & \frac{p-1}{2} & 0 \\
0 & 0 & 0 & \frac{-p}{2} & 0 & 1-\frac{p}{2} & 0 & 0 & \frac{p-1}{2} \\
0 & 0 & 0 & \frac{-p}{2} & \frac{-p}{2} & 0 & 1 & 0 & \frac{p-1}{2} \\
0 & 0 & 0 & 0 & -p & 0 & 0 & 1 & 0
\end{vmatrix}
$$

$$
= -\frac{(3p^3 - 16p^2 + 25p - 20)(p-1)^5}{64},
$$

$$
\widetilde{Q}_{12}(1,p) = \begin{vmatrix}
p-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1-\frac{p}{2} & \frac{p-1}{2} & \frac{p-1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{-p}{2} & 1-\frac{p}{2} & 0 & \frac{p-1}{2} & \frac{p-1}{2} & 0 & 0 & 0 & 0 \\
-p & 0 & 1 & 0 & p-1 & 0 & 0 & 0 & 0 \\
0 & \frac{-p}{2} & 0 & 1-\frac{p}{2} & 0 & \frac{p-1}{2} & \frac{p-1}{2} & 0 & 0 \\
0 & \frac{-p}{2} & \frac{-p}{2} & 0 & 1 & 0 & \frac{p-1}{2} & \frac{p-1}{2} & 0 \\
0 & 0 & 0 & \frac{-p}{2} & 0 & 1-\frac{p}{2} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{-p}{2} & \frac{-p}{2} & 0 & 1 & 0 & \frac{p-1}{2} \\
0 & 0 & 0 & 0 & -p & 0 & 0 & 1 & p-1
\end{vmatrix}
$$

$$
= \frac{(p^4 - 15p^3 + 53p^2 - 71p + 40)(p-1)^5}{64}.
$$

Therefore

$$
R_{10} = \frac{p^3 - p^2 + 4}{p^4 - 17p^3 + 68p^2 - 96p + 64},
$$

$$
R_{11} = -\frac{3p^3 - 16p^2 + 25p - 20}{p^4 - 17p^3 + 68p^2 - 96p + 64},
$$

$$
R_{12} = \frac{p^4 - 15p^3 + 53p^2 - 71p + 40}{p^4 - 17p^3 + 68p^2 - 96p + 64}.
$$

For $m = 6$ the Markov chain has 16 states and the quantities $R_{13}, R_{14}, R_{15}, R_{16}$ represent the limiting probability distribution of the absorbing states $\{0,6\}, \{1,5\}, \{2,4\}, \{3,3\}$. Similar calculations lead to

$$
R_{13} = -\frac{p^4 - 5p^3 + 4p^2 - 4}{2(p^6 - 9p^5 + 44p^4 - 120p^3 + 192p^2 - 160p + 64)},
$$

$$
R_{14} = \frac{3p^4 - 16p^3 + 41p^2 - 44p + 24}{2(p^6 - 9p^5 + 44p^4 - 120p^3 + 192p^2 - 160p + 64)},
$$

$$
R_{15} = \frac{p^6 - 9p^5 + 45p^4 - 125p^3 + 196p^2 - 160p + 60}{2(p^6 - 9p^5 + 44p^4 - 120p^3 + 192p^2 - 160p + 64)},
$$

$$
R_{16} = \frac{p^6 - 9p^5 + 41p^4 - 104p^3 + 151p^2 - 116p + 40}{2(p^6 - 9p^5 + 44p^4 - 120p^3 + 192p^2 - 160p + 64)}.
$$

As $p \to 1$ we obtain $R_{10} = 1/5, R_{11} = R_{12} = 2/5$ for $m = 5$ and $R_{13} = R_{16} = 1/6, R_{14} = R_{15} = 2/6$ for $m = 6$, which shows that in both cases the distribution of k_1 becomes uniform when p approaches unity.

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