Journal of Universal Computer Science, vol. 3, no. 3 (1997), 172-184
submitted: 28/3/96, accepted: 18/3/97, appeared: 28/3/97 *© Springer Pub. Co.*

Difference Splittings of Recursively Enumerable Sets

Asat Arslanov
(Computer Science Department
University of Auckland
P.O.Box 92019, Auckland
New Zealand
e-mail :aars01@cs.auckland.ac.nz)

Abstract. We study here the degree-theoretic structure of set-theoretical splittings of recursively enumerable (r.e.) sets into differences of r.e. sets. As a corollary we deduce that the ordering of **wtt**–degrees of unsolvability of differences of r.e. sets is not a distributive semilattice and is not elementarily equivalent to the ordering of r.e. **wtt**–degrees of unsolvability.

Keywords: Recursively enumerable sets, degrees of unsolvability, weak truth table reducibility.

1 Introduction

We review here the main notation and notions which will be used in this paper. All other notation and notions can be found in [27] and [28]. It is assumed that the reader is familiar with standard tree arguments (see [28]). *Recursively enumerable* (*r.e.*) sets are the sets for which there exist Turing machines that effectively enumerate them. The set of all natural numbers is denoted by ω. A set $A \subseteq \omega$ is called *d–r.e.* (*difference of r.e. sets*) if there are two r.e. sets of natural numbers $A_1, A_2 \subseteq \omega$ such that $A = A_1 - A_2$.

Let $\{W_e\}_{e\in\omega}$ and $\{\varphi_e\}_{e\in\omega}$ be the standard enumerations of recursively enumerable sets and partial recursive functions, respectively. We will denote partial recursive functionals (Turing reductions/Turing computations) by capital Greek letters $\Phi, \Psi, ,$, and sets of natural numbers and their corresponding characteristic functions by capital Latin letters. For sets A and B, put $A \oplus B = \{2x : x \in A\} \bigcup \{2x+1 : x \in B\}$. Here A_s is the finite part of the set A enumerated at stage s. Denote by $\Phi_{e,s}(A_s, x)\downarrow$ the fact that the partial recursive (p.r.) functional with oracle A_s converges in s stages on the input x; $\Phi_{e,s}(A_s, x)\uparrow$ denotes divergence (i.e. there is no outcome of computation) at stage s. The function $\lambda x, y\langle x, y\rangle$ denotes a pairing of $\omega \times \omega$, i.e. a recursive bijection from $\omega \times \omega$ onto ω. Using this mapping one inductively gets computable coding of all n–tuples of numbers. The restriction of the set/function A to the initial segment of length $k+1$ is denoted by $A\lceil k+1 = \{x \in A : x \leq k\}$. For sets $A, B \subseteq \omega$, A is *Turing reducible* (*T–reducible*) to B, denoted by $A \leq_T B$, if there is an $e \in \omega$ such that for all x, $\Phi_e(B;x) = A(x)$. The use–function for $\Phi_e(A,x)$ is defined as follows:

$$use(\Phi_e(A,x)) = \begin{cases} \mu y[\Phi_e(A\lceil y+1;x)\downarrow = \Phi_e(A;x)\downarrow], \\ \textbf{undefined}, \text{ otherwise.} \end{cases}$$

Here we use the standard μ notation for the minimization operator. As usual we assume that the use–function has the property that for all e, s, A, x if $\Phi_{e,s}(A_s;x)\downarrow$ then $e, x, use(\Phi_{e,s}(A_s;x)) < s$. The set A is *weakly truth table reducible* to B, denoted by $A \leq_{wtt} B$, if there exist $e_0, e_1 \in \omega$ such that for all x , $\Phi_{e_0}(B;x) = A(x)$, $\phi_{e_1}(x)\downarrow$ and $use(\Phi_{e_0}(B;x)) \leq \phi_{e_1}(x)$, that is, A is Turing reducible to B and the use–function of the Turing reduction is majorised by some total recursive function. We use here *wtt*–functionals defined as follows. Let $\{(\Phi_e, \phi_e)\}_{e\in\omega}$ be standard enumeration of all pairs of partial recursive (p.r.) functionals and p.r. functions. Then define

$$\widehat{\Phi}_e(A;x) = \begin{cases} \Phi_e(A;x)\downarrow \text{ and } use(\Phi_e(A;x)) \leq \phi_e(x)\downarrow, \\ \textbf{undefined}, \text{ otherwise.} \end{cases}$$

The $\widehat{\Phi}_{e,s}(A;x)$–computation of the *wtt*–functional, executed in s stages, is defined analogously. It is clear that $A \leq_{wtt} B$ is equivalent to $\widehat{\Phi}_e(B) = A$ for some e. From now on we omit the superscript symbols and that of the stage s when from the context it will be clear that we deal with *wtt*–functionals and computations at stage s. We will say that the set A $wtt - (T-)$ *computes* the set B if $B \leq_{wtt} A$ $(B \leq_T A)$.

Equivalence classes induced by these reducibility relations are called **T**– and **wtt**–*degrees of unsolvability*. The **T**–degree (sometimes called *Turing* degree) of A is denoted by the corresponding bold Latin letter **a** or $deg(A)$, and the **wtt**–degree of A by the corresponding bold capital Latin letter. A degree of unsolvability is called recursively enumerable (d–r.e.) if it contains an r.e. (d–r.e.) set.

Now we review some facts about sets T–reducible to the Halting Problem set $\emptyset' = \{x : \varphi_x(x)\downarrow\}$. First we note here the *Limit Lemma* (see [28]):

$A \leq \emptyset'$ if and only if there is some recursive function g such that

$$A(x) = lim_s g(x,s).$$

Here $lim_s g(x,s) = A(x)$ means that there exists t such that for all $s \geq t$ $A(x) = g(x,s)$. When we have the function $g(x,s)$ as in this definition, we put $A_s(x) = g(x,s)$ and call it a *recursive approximation to A*.

This characterization of the sets T–reducible to the Halting Problem (or equivalently, T–computable by the Halting Problem) naturally suggests a possible hierarchy for them. This hierarchy is based on the approach to classify a set by measuring how often an approximation to the characteristic function of a set changes before it stabilizes.

For example, there exists another equivalent way to define r.e. (d–r.e.) sets, which is by recursive approximation of their characteristic functions with at most one (two) change in the approximation: for a given set A we start by guessing that x is not in A and we may change our guess about the membership of x at most once in the r.e. case and twice in the d–r.e case, namely when we enumerate x into A and when we extract it from A. If one allows the approximation to change more often then this approach leads to the definition of a concept of an *n–recursively enumerable* set which includes the definitions for the r.e. and d–r.e. sets as particular cases.

A set $A \subseteq \omega$ is called *n–recursively enumerable (n–r.e.)* if there is a recursive function f such that for all x:

1. $A(x) = \lim_s f(x,s)$,
2. $f(x,0) = 0$,
3. $|\{s : f(x,s) \neq f(x,s+1)\}| \ \leq \ n$.

The class of all r.e. sets coincides with the class of 1–r.e. sets, and the class of differences of r.e. sets coincides with the class of 2–r.e. sets. A recursive enumeration of an infinite r.e. set is denoted by $\{A_s\}_{s\in\omega}$, where $|A_{s+1} - A_s| = 1$ and $\{a_s\} = A_{s+1} - A_s$; for a finite set X, $|X|$ denotes the cardinality of X. The same notation is used for a recursive approximation of a d–r.e. set A with the property that for all x $|\{s : A_s(x) \neq A_{s+1}(x)\}| \ \leq \ 2$.

Let us notice the well known fact (see [18]) that for all natural numbers n the class of n–r.e. sets coincides with the class of all sets obtained by closure of the r.e. sets under complementation, union and intersection (i.e. the smallest Boolean algebra generated by r.e. sets). It could be easily shown that a set is n–r.e. if and only if it is a finite union of d-r.e. sets. We should note that the class of sets T–reducible to $\emptyset'$ is not exhausted by the hierarchy of sets whose approximation may change only n times. A transfinite extension of the hierarchy (see [18]) describes, however, all these sets.

The hierarchy of recursively approximated sets was first introduced and studied by Putnam (see [26]) and Ershov (see [18]). The Turing degrees of n–r.e. sets were first studied by Cooper and Lachlan (see [19]). Cooper proved (see [5]) that the n–r.e. sets generate a proper degree hierarchy below $\mathbf{0}'$, the degree of Halting Problem, that is, for each $n \geq 1$, there is an $(n+1)$–r.e. set such that its degree does not contain any n–r.e. set.

The set of all n–r.e. **wtt**– and Turing degrees is denoted by $D_{n,wtt}$ and D_n, respectively. $\mathbf{D_{n,wtt}} \overset{\text{def}}{=} \langle D_{n,wtt}; \leq, \bigcup, \bigcap \rangle$ denotes the partial ordering of n–r.e.

$\mathbf{wtt}$–degrees. In $\mathbf{D_{n,wtt}}$ one can naturally define the operation of least upper bound and the partial operation of greatest lower bound.

Weak truth table reducibility (*wtt*–reducibility) has been studied in the theory of recursive functions for a long time (it was introduced by Friedberg and Rogers, see [20]) and turned out to be an important concept in the investigations of the lattice of r.e. sets and the algebraic structure of partial ordering of r.e. Turing degrees (see [2, 3, 4, 9, 11, 13, 14, 15, 17, 23, 24, 29]). This notion is useful in effective algebra, where, for example, Downey and Remmel used this notion to solve the classification problem of the algorithmic complexity of r.e. bases of r.e. vector spaces (see [14, 15]). In fact they proved that r.e. $\mathbf{wtt}$–degrees below (in the ordering induced by *wtt*–reducibility) the $\mathbf{wtt}$–degree of the given vector space V are exactly $\mathbf{wtt}$–degrees of r.e. bases of this space V.

In this paper we study the degree–theoretic structure (under *wtt*–reducibility) of d–r.e. splittings of r.e. sets.

Definition 1.1. *A* **difference (r.e.) splitting** *of an r.e. set A is a pair of d–r.e. (r.e.) sets D_1 and D_2 such that $D_1 \bigcup D_2 = A$ and $D_1 \bigcap D_2 = \emptyset$, where $\bigcup$ and $\bigcap$ are standard set–theoretic operations.*

It turns out that the degree–theoretic structure of difference splittings of r.e. sets is quite different from the degree–theoretic structure of r.e. splittings of r.e. sets. For example, for every r.e. splitting A_1, A_2 of any given r.e. set A, $A_i \leq_T A, i = 1, 2$ and $A_1 \oplus A_2 \equiv_T A$, while there are difference splittings wich do not possess these properties of r.e. splittings. For example, for any not T-complete r.e. set A the pair of d-r.e. sets A_0 and A_1, where A_0 is T-complete r.e. subset of A (there always exists one) and $A_1 = A - A_0$, is such a difference splitting.

A number of properties simultaneously true for all semilattices $\mathbf{D_{n,wtt}}, \mathbf{n} < \omega$ have been found. For example:

1. (Ladner–Sasso, [23]) Density and splitting hold simultaneously in the r.e. $\mathbf{wtt}$–degrees, i.e. the following statement

$$(\forall\ \mathbf{A}, \mathbf{B})(\mathbf{A} < \mathbf{B} \implies (\exists\ \mathbf{C_0}, \mathbf{C_1})(\mathbf{A} < \mathbf{C_0}, \mathbf{C_1} < \mathbf{B}\ \wedge\ \mathbf{C_0} \cup \mathbf{C_1} = \mathbf{B}))$$

holds true in the algebraic structure $\mathbf{D_{1,wtt}}$.

1′. (see [1]) For a given $n \geq 2, n \in \omega$, density and splitting hold true simultaneously in $\mathbf{D_{n,wtt}}$.

2. (Ladner–Sasso, [23]) Anticupping property holds for every nonrecursive r.e. $\mathbf{wtt}$–degree, i.e. the statement

$$(\forall\,\mathbf{A})(\mathbf{A} > \mathbf{0} \implies (\exists \mathbf{B} < \mathbf{A})(\forall \mathbf{C})(\mathbf{B} \cup \mathbf{C} \geq \mathbf{A} \Rightarrow \mathbf{B} \geq \mathbf{A}))$$

holds true in $\mathbf{D_{1,wtt}}$.

2′. (Downey, [9]) Strong anticupping property holds for every nonrecursive r.e. $\mathbf{wtt}$–degree, i.e. (the notation in the statement shows that the first two quantifiers range through $\mathbf{D_{1,wtt}}$ and the third one ranges through $\mathbf{D_{wtt}}(\leq \mathbf{0'_{wtt}}))$

$$(\forall\ \mathbf{A}\ r.e.)(\mathbf{A} > \mathbf{0} \implies (\exists\, \mathbf{B}\ r.e.\ < \mathbf{A})(\forall \mathbf{C}\ \Delta^2_0)(\mathbf{B} \cup \mathbf{C} \geq \mathbf{A} \Rightarrow \mathbf{B} \geq \mathbf{A}))$$

3. (Cohen, [6]) Every non wtt–complete r.e. **wtt**–degree is branching both in r.e. **wtt**–degrees and in n–r.e. **wtt**–degrees, for any $n \geq 2$.

Notice that the first two statements are among the most interesting structural properties (e.g. density/nondensity for partial orderings) that prove the elementary non–equivalence of the partial orderings of r.e **T**–degrees and d–r.e. **T**–degrees (see, for example, [7]). All these facts point out to the existence of a great similarity in the structure of partial orderings of **wtt**–degrees. Nevertheless, in the next paragraph it is proved that $\mathbf{D_{n,wtt}}$ is a nondistributive semilattice, while it was shown by Lachlan (see [29]) that the partial ordering of r.e. **wtt**–degrees forms a distributive semilattice.

2 Embedding of a Nondistributive Lattice Into d–R.E. wtt–Degrees

For every non **wtt**–complete set A we construct an r.e. set E and a difference splitting D_0, D_1 of E such that the wtt-degrees of the sets $E \oplus A, D_0 \oplus A, D_1 \oplus A, D_0 \oplus D_1 \oplus A, A$ form an isomorphic copy of the nondistributive five-element modular lattice M_5 (see [21]). Since $\mathbf{D_{1,wtt}}$ is known to be nondistributive, it follows that for every $n \geq 2$ the partial orderings $\mathbf{D_{n,wtt}}$ and $\mathbf{D_{1,wtt}}$ are not elementarily equivalent.

Theorem 2.1. *For every non* **wtt**–*complete set A there exist an r.e. set E and a difference splitting D_0, D_1 of the set E such that the* **wtt**–*degrees of the sets $E \oplus A, D_0 \oplus A, D_1 \oplus A, D_0 \oplus D_1 \oplus A, A$ constitute the lattice–theoretic embedding of the modular lattice $\mathbf{M_5}$ into the upper semilattice of n–r.e.* **wtt**–*degrees, for any fixed $n \geq 2$.*

Proof. We will construct an r.e. set E and its difference splitting D_0, D_1 satisfying the requirements: one global set–theoretic requirement $\mathcal{P}$

$$\begin{cases} x \in E_{s+1} \backslash E_s \longrightarrow x \in D_{0,s+1} \text{ or } D_{1,s+1}, \\ x \in D_{i,s} \backslash D_{i,s+1} \longrightarrow x \in D_{1-i,s+1} \backslash D_{1-i,s}; \end{cases}$$

and the infinite linearly ordered list of requirements:

$$\begin{array}{rl} \mathcal{P}_e : & E \neq \Phi_e(A)\ ; \\ \mathcal{N}_e : & \Phi_e(D_0 \oplus A) = \Phi_e(D_1 \oplus A) = f \ \text{ total function } \implies f \leq_{wtt} A\,; \\ \mathcal{NP}_{\langle e,i \rangle} : & \Phi_e(E \oplus A) = \Phi_e(D_i \oplus A) = f \ \text{ total function } \implies f \leq_{wtt} A, \\ & \text{where } i = 0, 1. \end{array}$$

Notice that we are using Posner's lemma (see [28], p.153) which says that it suffices for our purposes to satisfy the negative requirements $\mathcal{N}_e$ and $\mathcal{NP}_{\langle e,i\rangle}$ with the same e on both sides of their antecedents.

Lemma 2.2. *The* $\mathbf{wtt}$*–degrees of the sets* $A, E\oplus A, D_0\oplus A, D_1\oplus A, D_0\oplus D_1\oplus A$ *that satisfy the above list of requirements* $\mathcal{P}, \mathcal{P}_e, \mathcal{N}_e, \mathcal{NP}_e, e\in\omega$ *constitute a lattice–theoretic embedding of the lattice* $\mathbf{M_5}$ *into the upper semilattice* $\mathbf{D_{n,wtt}}$*, for any fixed* $n\geq 2$.

Proof. 1. $E\oplus A\leq_{wtt} D_0\oplus D_1\oplus A$, because $E = D_0\bigcup D_1$ and $D_0\bigcap D_1=\emptyset$. It is clear that $D_i\oplus E\leq_{wtt} D_0\oplus D_1$.

2. $D_0\oplus D_1\leq_{wtt} D_i\oplus E$. It is sufficient to show that $D_i\leq_{wtt} D_{1-i}\oplus E$. Let us compute $D_i(x)$ for an arbitrary x. First we query the oracle $E: 2x+1\in E$? If the answer is a positive one then we question the oracle $D_{1-i}: 2x\in D_{1-i}$? If again we get a positive answer then $x\notin D_i$; if the answer is negative then $x\in D_i$. If $2x+1\notin E$ then it is obvious that $x\notin D_i$. Thus $D_0\oplus D_1\oplus A\equiv_{wtt} D_i\oplus E\oplus A, i=0,1$.

3. Certainly, no $D_i\oplus A, i=0,1$, *wtt*–computes the set $E\oplus A$, since otherwise the $\mathcal{NP}$–requirements would imply $E\leq_{wtt} A$.

4. $E\oplus A$ does not *wtt*–compute either of $D_i\oplus A, i=0,1$. We consider two cases. First let us suppose that $E\oplus A$ *wtt*-computes only one $D_i\oplus A$: $D_i\oplus A\leq_{wtt} E\oplus A$, but $D_{1-i}\not\leq_{wtt} E\oplus A$ for some $i=0,1$. $D_i\oplus E\oplus A\leq_{wtt} E\oplus A$ and $D_{1-i}\oplus A\not\leq_{wtt} E\oplus A$ imply that $D_{1-i}\oplus E\oplus A\not\leq_{wtt} D_i\oplus E\oplus A$, a contradiction to 2. Secondly, if we assume that $E\oplus A$ computes both sets D_i, $D_0\oplus A\leq_{wtt} E\oplus A$ and $D_1\oplus A\leq_{wtt} E\oplus A$, then it follows from the satisfaction of the $\mathcal{N}$–requirements that

$$D_0\oplus A\leq_{wtt} D_0\oplus A, E\oplus A\Longrightarrow D_0\oplus A\leq_{wtt} A$$

and

$$D_1\oplus A\leq_{wtt} D_1\oplus A, E\oplus A\Longrightarrow D_1\oplus A\leq_{wtt} A\Longrightarrow E\leq_{wtt} A,$$

since $E\leq_{wtt} D_0\oplus D_1$, which is a contradiction with conditions $\mathcal{P}_e, e\in\omega$. □

The requirements $\mathcal{P}_e$ will be satisfied by the modified Friedberg–Muchnik strategy and the requirements $\mathcal{N}_e$ by the modified minimal pair strategy. Let us describe the main module of the strategy for $\mathcal{NP}$–requirements. It will consists of two strategies — the standard strategy of minimal pair and the variant of Downey's strategy from the Diamond theorem ([12]). It could be the result of the joint work of the Friedberg–Muchnik strategy and of the attempt to satisfy the set–theoretic requirement about the splitting of the constructing set E that we should enumerate numbers simultaneously into E and the one D_i for some i. These actions would possibly destroy simultaneously both computations of the p.r. functionals $\Phi_e(E\oplus A;[l(\langle e,i\rangle,s)-1])$ and

$\Phi_e(D_i \oplus A;[l(\langle e,i\rangle,s)-1]$ for some requirement $\mathcal{NP}_{\langle e,i\rangle}$. Let, for example, $x \in E_{p+1}\backslash E_p$ and $x \in D_{i,p+1}\backslash D_{i,p}$ and $x < r(\langle e,i\rangle,p+1)$. Then at the first $\langle e,i\rangle$–expansionary stage, if it exists at all, $s+1 > p+1$, it is possible that the computations of both p.r. functionals for $\mathcal{NP}$–requirement will have different outcomes, and the length of agreement between them will be increased. That is, for some $y < l(\langle e,i\rangle, ls(\langle e,i\rangle,p+1)) : \Phi_{e,s+1}(E\oplus A;y) \neq \Phi_{e,ls(\langle e,i\rangle,p+1)}(E\oplus A;y)$ and $\Phi_{e,s+1}(D_i \oplus A;y) \neq \Phi_{e,ls(\langle e,i\rangle,p+1)}(D_i \oplus A;y)$. In this case the strategy for the requirement $\mathcal{NP}_{\langle e,i\rangle}$ becomes active and achieves an inequality at the stage $s+1$ by the transfering the number x from the set D_i into the D_{1-i}. It will restore its computation with oracle $D_i \oplus A$, that is,

$$\Phi_{e,s+1}(D_i \oplus A;y) = \Phi_{e,ls(\langle e,i\rangle,p+1)}(D_i \oplus A;y) =$$

$$= \Phi_{e,ls(\langle e,i\rangle,p+1)}(E \oplus A) \neq \Phi_{e,s+1}(E \oplus A;y).$$

To preserve the inequality we are not going to change the oracle $E \oplus A$ at the initial segment of length $\varphi_e(y)+1$.

Using the techniques of the priority method, all the above mentioned strategies easily cohere with each other with the one exception, which we will consider separately. The exception recurs when some $\mathcal{NP}_{\langle e,i\rangle}$–strategy α with finite outcome is situated on the tree of strategies below some $\mathcal{N}_j$–strategy or $\mathcal{NP}_{\langle k,l\rangle}$–strategy with an infinite outcome, that is, $\widehat{\beta\langle 0\rangle} \subseteq \alpha$ (according to the notation we introduce in the next paragraph). Let us suppose that at some stage $s+1$ the following situation holds for some $x < l(e,s+1):\ x \in E_{s+1}\backslash E_s$ and $x \in D_{i,s+1}$, and

$$\Phi_{e,s+1}(D_0 \oplus A;x) = \Phi_{e,s+1}(D_1 \oplus A;x) = q,$$

and at all e–expansionary stages the infinite outcome of the requirement $\mathcal{N}_e$ depends on x remaining in D_i. If at some stage $t+1$ $\mathcal{NP}$–strategy α becomes active with this number x: $x \in D_{1-i,t+1}\backslash D_{1-i,t}$ and $x \notin D_{i,t+1}$, then the corresponding $\mathcal{N}$–strategy β could be injured by the changes to both oracles. Therefore at the next e–expansionary stage $u+1$ we should check whether the computations of p.r. functionals in the requirement $\mathcal{N}_e$ are different: $\Phi_{e,u+1}(D_i \oplus A;x) = q$? If they are different then we construct wtt–reduction $\Phi(A) = f$.

In this construction we use the tree of strategies denoted by $T = \{0,1\}^{<\omega}$. As often in an infinite injury construction (see [28]), we fix the set of outcomes $\{0,1\}$ with the usual order, so that the infinite outcome of strategy is denoted by 0, and the finite by 1. We assign to all nodes of $\alpha \in T$ of length $3e$ the requirement $\mathcal{P}_e$ (sometimes we will call such nodes $\alpha(\mathcal{P}_e)$), to all nodes of length $3e+1$ the requirement $\mathcal{NP}_e$, and to all nodes of length $3e+2$ the requirement $\mathcal{N}_e$. For α corresponding to $\mathcal{P}_e, \mathcal{N}_e$ and $\mathcal{NP}_e$ we are using the following auxiliary length of agreement functions:

$$lp(\alpha, s) = max\{x : (\forall y < x)(\Phi_{e,s}(A_s; y) = E_s(y))\};$$

$$l(\alpha, s) = max\{x : (\forall y < x)(\Phi_{e,s}(D_{0,s} \oplus A_s; y) = \Phi_{e,s}(D_{1,s} \oplus A_s; y))\};$$

$$ml(\alpha, s) = max\{l(\alpha, t) : t < s \text{ and } t \text{ is } \alpha - \text{stage}\};$$

$$L(\alpha, s) = max\{x : (\forall y < x)(\Phi_{e,s}(E_s \oplus A_s; y) = \Phi_{e,s}(D_{i,s} \oplus A_s; y))\};$$

$$M(\alpha, s) = max\{L(\alpha, t) : t < s \text{ and } t \text{ is } \alpha - \text{stage}\};$$

$$ls(\alpha, s) = max\{0, t : t < s \text{ and } l(\alpha, t) > ml(\alpha, t)\};$$

We recall that the stage $s+1$ is called α–expansionary if it is an α–stage (see [28]) and $l(\alpha, s+1) > ml(\alpha, s+1)$. Here under l and ml we mean the length of agreement functions for the corresponding α. For every strategy α we fix the standard enumeration of the creative set K at the α–expansionary stages, so that the k-th element of the set K is enumerated at the k-th α-expansionary stage.

Construction. At stage 0 all the strategies are initialized, i.e. they are in the state when all parameters (if they are assigned) and computations are declared undefined.

Stage $s+1$. Define the approximation to the so called true path f (see [28, Chapter 14]) $\delta_{s+1} : |\delta_{s+1}| \leq s$. Let $\delta_{s+1}\lceil 0 = \emptyset$. Let we already have defined $\delta_{s+1}\lceil(n) = \alpha$. Now we define $\delta_{s+1}(n)$ by following the stated below conditions.

If $|\alpha| = 3e$ for some e, then execute the corresponding action.

1. The strategy α does not have an assigned number. If the stage $s+1$ is the $k+1$-th α–expansionary stage then assign the number $x_\alpha \stackrel{\text{def}}{=} \langle c(\alpha), x_{k+1}\rangle$ as a *witness* of the strategy. Here $x_{k+1} \in K_{k+1}$ and $c(\alpha)$ is a code of the node α in the fixed numbering of all finite binary sequences. Initialize all $\xi > \alpha$ and finish the stage.

2. For some witness $x_\alpha : \Phi_{e,s+1}(A_{s+1}; x_\alpha) \downarrow= 0$ and $E_{s+1}(x_\alpha) = 1$. Then put $\delta_{s+1}\lceil(n) = 0$.

3. For some witness x_α, $\Phi_{e,s+1}(A_{s+1}; x_\alpha) \downarrow= 0$ and $E_{s+1}(x_\alpha) = 0$. Then put $x_\alpha \in E_{s+1} \backslash E_s$. Initialize all $\xi > \alpha$ and finish the stage.

4. For some assigned witness $x_\alpha : \Phi_{e,s+1}(A; x_\alpha) \uparrow$. Then let $\delta_{s+1}(n) = 1$.

If $|\alpha| = 3e+1$ and for some $e : e = \langle i, sg(j)\rangle$, where

$$sg(x) = \begin{cases} 1, x \geq 1; \\ 0, x = 0. \end{cases}$$

1. Stage $s+1$ is not α–expansionary. Then $\delta_{s+1}(n) = 1$.

2. Some strategy $\beta(P'_e) : \widehat{\alpha\langle 0\rangle} \subseteq \beta$ executed at some preceding α–expansionary stage $u+1$ point 3 with the witness x_β and for some $y < l(\alpha, u+1)$:

$$\Phi_{i,s+1}(E \oplus A; y) \neq \Phi_{i,u+1}(E \oplus A; y)$$

and

$$\Phi_{i,s+1}(D_{sg(j)} \oplus A; y) \neq \Phi_{i,u+1}(D_{sg(j)} \oplus A; y).$$

Then enumerate the number x_β from the set $D_{sg(j)}$ into $D_{1-sg(j)}$. Initialize all $\xi > \alpha$ and finish the stage.

3. In the case opposite to the previous two define $\delta_{s+1}(n) = 0$.

Let $|\alpha| = 3e+2$:

1. Stage $s+1$ is not α–expansionary. Then $\delta_{s+1}(n) = 1$.

2. Some strategy $\beta(P_{e'}) : \widehat{\alpha\langle 0\rangle} \subseteq \beta''$ fulfilled at stage $ls(\alpha, ls(\alpha, s+1))$ point 3 with the witness x_β, some strategy $\beta'(\mathcal{NP}_{e''}) : \widehat{\alpha\langle 0\rangle} \subseteq \widehat{\beta'\langle 0\rangle} \subseteq \beta''$ fulfilled point 2 at stage $ls(\alpha, s)$, and for some $y < ls(\alpha, s+1)$, where $s+1$ is the k–th α–expansionary stage:

$$\Phi_{e,s+1}(D_0 \oplus A; y) \neq \Phi_{e,ls(\alpha,s+1)}(D_0 \oplus A; y) \text{ and}$$
$$\Phi_{e,s+1}(D_1 \oplus A; y) \neq \Phi_{e,ls(\alpha,s+1)}(D_1 \oplus A; y).$$

Then enumerate the number $\langle e, y, k, \varphi_e(y)\rangle$ in E_{s+1}. Initialize all $\xi > \alpha$ and finish the stage.

3. In the case opposite to the preceding two cases define $\delta_{s+1}(n) = 0$.

Initialize all $\xi : \alpha <_L \xi$.

The end of stage $s+1$.

The *true path* f is defined by induction as follows: $f\lceil 0 = \emptyset$. If $f\lceil n$ is defined then

$$f(n) = \mu\{k : k \in \{0,1\} \ \& \forall s\, \exists t > s\, \widehat{f\lceil n\langle k\rangle} \subseteq \delta_t\}.$$

Now let us show that the function $\lambda n f(n)$ is defined everywhere and the strategy $f\lceil n$ satisfies the corresponding requirement.

Lemma 2.3. *For all positive integers n, $f\lceil n$ does exist and acts at most finitely many times in the construction. If $f\lceil n = \alpha$ is defined and is $\mathcal{N}$– or $\mathcal{NP}$–strategy with finite outcome, or $\mathcal{P}$–strategy, then the corresponding requirement is satisfied.*

Proof. By the definition $f\lceil 0 = \emptyset$. The induction step: we assume that the statement of the Lemma holds for $\alpha = f\lceil m$, for $m < n$ and fix the least α–stage s after which α will never be initialized.

Let $|\alpha| = 3e+1$. Let us suppose that $\lim_s L(\alpha, s+1) = \infty$ since otherwise the statement is obvious. Let us suppose that α acts after stage s; let $t+1$ be the least such stage. Then some $\mathcal{P}$–strategy β acted at the preceding α–expansionary stage and at the first after s α–expansionary stage $t+1$ for some $y < L(\alpha, t+1)$: $\Phi_{i,s+1}(E \oplus A; y) \neq \Phi_{i,t+1}(E \oplus A; y)$. Then the strategy $\alpha(\mathcal{NP})$ acts by enumerating the number x_β from $D_{sg(j)}$ into $D_{1-sg(j)}$ and restores the oracle $(D_{sg(j),t+1} \oplus A_{t+1})\lceil \varphi_e(y) = (D_{sg(j),s+1} \oplus A_{s+1})\lceil \varphi_e(y)$, and achieves the inequality at stage $t+1$. In this case $\delta\lceil(n+1) = \widehat{\alpha\langle 1\rangle}, \mathcal{NP}_e$ is met and $\alpha(\mathcal{NP})$ will not be injured and will not act anymore.

If $|\alpha| = 3e$, that is, α is $\mathcal{P}$–strategy. Let us suppose that the corresponding requirement is not satisfied, i.e. $\lim_s lp(\alpha, s) = \infty$. This means that for some number z the following statement holds true:

$$(\forall x)(x > z \Longrightarrow ((x \in K \Longleftrightarrow (\exists t)(\langle c(\alpha), x\rangle \in E_{t+1} \backslash E_t)) \Longleftrightarrow$$

$$\Longleftrightarrow (x \in K \Longleftrightarrow (\exists s)(a_s < \phi_e(\langle c(\alpha), x\rangle) \text{ and } a_s \in A_{s+1} \backslash A_s))) \Longrightarrow K \leq_{wtt} A.$$

Hence there a stage u such that α executes the point 3 at this stage, i.e. $E(x_\alpha) = 1 \neq 0 = \Phi_e(A; x_\alpha)\downarrow$, and afterwards, by the assumption, the strategies of higher priority do not act anymore and α initializes all $\xi > \alpha$ at stage u. Therefore for every α–stage $v > u$ α is in the state 2 and $\delta\lceil(n+1) = \widehat{\alpha\langle 0\rangle}$. The case when $|\alpha| = 3e+2$ is also obvious. □

Lemma 2.4. *Let* $\widehat{\alpha 0} \subset f$ *for* $|\alpha| = 3e+1, 3e+2$. *Then the requirements* $\mathcal{N}_e$ *and* $\mathcal{NP}_e$ *are satisfied.*

Proof. By the preceding lemma we can fix the least α–stage s such that α will neither be initialized nor be active after stage s, since otherwise it would be that $\widehat{\alpha 1} \subset f$. Consequently, $lim_s\, l(\alpha, s) = \infty$. Let us fix arbitrary $x \in \omega$. Let s be the least stage $s > s_0$: s is α–expansionary and $l(\alpha, s) > x$, and

$$A_s\lceil\langle e, x, 2\phi_e(x), \phi_e(x)\rangle = A\lceil\langle e, x, 2\phi_e(x), \phi_e(x)\rangle.$$

Let $\Phi_{e,s_1}(D_{0,s_1} \oplus A_{s_1}; x) = \Phi_{e,s_1}(D_{1,s_1} \oplus A_{s_1}; x) = p$ and let $s_1 < s_2 < \ldots < s_n < \ldots$ are α–expansionary stages greater than s_1. Then

$$(\forall n)[\Phi_{e,s_n}(D_{0,s_n} \oplus A_{s_n}; y) = \Phi_{e,s_n}(D_{1,s_n} \oplus A_{s_n}; y) = p]$$

and $\Phi_e(D_i \oplus A; y) = p$, $i = 0, 1$. Notice that the numbers enumerated into A and $D_i, i = 0, 1$, could injure only one side of the equation, because the changes of both sides are coded into A and there exist at most $2\varphi_e(x)$ changes in A which could make such injuries. □

□

Corollary 2.5. *For every incomplete r.e.* **wtt**–*degree* **A** *there exists a zero preserving lattice theoretic embedding of the modular non-distributive lattice* $\mathbf{M_5}$ *into* $\mathbf{D_{2,wtt}}(\geq \mathbf{A})$.

Corollary 2.6. *For every incomplete r.e.* **wtt**–*degree* **A** *the partial ordering* $\mathbf{D_{2,wtt}}(\geq \mathbf{A})$ *does not form a distributive semilattice.*

Corollary 2.7. *For all positive integers* $n \geq 2$ *and for every incomplete r.e.* **wtt**–*degree* **A**, *the partial orderings* $\mathbf{D_{n,wtt}}(\geq \mathbf{A})$ *and* $\mathbf{D_{1,wtt}}(\geq \mathbf{A})$ *are not elementarily equivalent.*

The question remains if the structures $\mathbf{D_{n,wtt}}$ are all pairwise elementarily inequivalent for $n \geq 1$. The existence of many results which hold true simultaneously for all these structures with $n \geq 2$ suggests the following interesting conjecture: all the partial orderings $\mathbf{D_{n,wtt}}$ for $n \geq 2$ are pairwise elementarily equivalent.

Acknowledgement. I express my gratitude to Professor C.Calude and the anonymous referees for their comments.

References

[1] A.Arslanov, Partial orderings of n–r.e. **wtt**–degrees, Technical Report, VINITI, Moscow, No. 93–54.

[2] K.Ambos-Spies, Anti–mitotic recursively enumerable sets, *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik* 31 (1985), 461–467.

[3] K.Ambos-Spies and R.I.Soare, The recursively enumerable degrees have infinitely many one types, *Annals of Pure and Applied Logic* 44 (1989), 1–23.

[4] K.Ambos-Spies and P.A.Fejer, Degree theoretic splitting properties of recursively enumerable sets, *J. Symbolic Logic* 53 (1988), 1110–1137.

[5] S.B.Cooper, *Degrees of Unsolvability*, Ph.D. Thesis, Leicester University, Leicester, England, 1971

[6] P.F.Cohen, *Weak truth table reducibility and pointwise ordering of 1–1 recursive functions*, Ph.D.Thesis, University of Illinois at Urbana–Champaign, 1975.

[7] S.B.Cooper, L.Harrington, A.H.Lachlan, S.Lempp, and R.I.Soare, The d–r.e. degrees are not dense, *Annals of Pure and Applied Logic* 55 (1993), 125–151.

[8] R.G.Downey, The degrees of r.e. sets without universal splitting property, *Trans. Amer. Math. Soc.* 291 (1985), 337–351.

[9] R.G.Downey, Δ_0^2 Degrees and transfer theorems, *Illinois J. Math.* 31 (1987), 419–427.

[10] R.G.Downey, Subsets of hypersimple sets, *Pacific J. Math.* 127 (1987), 299–319.

[11] R.G.Downey, Intervals and sublattices of the r.e. weak truth table degrees, part 1: Density, *Annals of Pure and Applied Logic* 41 (1989), 1–26.

[12] R.G.Downey, D.r.e. degrees and the nondiamond theorem, *Bulletin of London Mathematical Society* 21 (1989), 43–50.

[13] R.G.Downey and C.G.Jockusch, T–degrees, jump classes and strong reducibilities, *Trans. Amer. Math. Soc.* 30 (1987), 103–137.

[14] R.G.Downey and J.B.Remmel, Classification of degree classes associated with r.e. subspaces, *Annals of Pure and Applied Logic* 42 (1989), 105–125.

[15] R.G.Downey, J.B.Remmel and L.V.Welsh, Degrees of splittings and bases of recursively enumerable subspaces, *Trans. Amer. Math. Soc.* 302 (1987), 683–714.

[16] R.G.Downey and M.Stob, Automorphisms of the lattice of recursively enumerable sets: Orbits, *Advances in Math.* 92 (1992), 237–265.

[17] R.G.Downey and L.V.Welsh, Splitting properties of r.e.sets and degrees, *J. Symbolic Logic* 51 (1986), 88–109.

[18] Yu.L.Ershov, On a hierarchy of sets I, *Algebra i Logika* 1 (1968), 47–73.

[19] R.L.Epstein, *Degrees of Unsolvability: Structure and Theory*, Lecture Notes in Mathematics No. 759, Springer–Verlag, Berlin, Heidelberg, New-York, 1979.

[20] R.Friedberg and H.Rogers, Jr., Reducibilities and completeness for sets of integers, *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik* 5 (1959), 117–125.

[21] G.Grätzer, *General lattice theory*, Birkhäuser, Basel and Academic Press, New York, 1978.

[22] D.Kaddah, Infima in the *d*–r.e. degrees, *Annals of Pure and Applied Logic* 62 (1993), 207–263.

[23] R.Ladner and L.Sasso, The weak truth table degrees of recursively enumerable sets, *Annals of Mathematical Logic* 8 (1975), 429–448.

[24] A.H.Lachlan, The priority method I, *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik* 13 (1967), 1–10.

[25] A.H.Lachlan, Decomposition of recursively enumerable degrees, *Proc. Amer. Math. Soc.* 79 (1980), 629–634.

[26] H.Putnam, Trial and error predicates and the solution to a problem of Mostowski, *J. Symbolic Logic* 30 (1965), 49–57.

[27] H.Rogers, Jr. *Theory of Recursive Functions and Effective Computability*, McGraw–Hill, New York, 1967.

[28] R.I.Soare, *Recursively Enumerable Sets and Degrees*, Springer–Verlag, Berlin, 1987.

[29] M.Stob, **Wtt**–degrees and **T**–degrees of r.e. sets, *J. Symbolic Logic* 48 (1983), 921–930.