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Effectiveness of the Completeness Theorem for an Intermediate Logic[1]

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Abstract: We investigate effectiveness of the completeness result for the logic with the Weak Law of Excluded Middle.

Key Words: computability, Kripke models, completeness, jump operator, intermediate logics.

Category: F.1, F.4

1 Motivation

Investigations of effectiveness in intuitionistic model theory can be considered as a part of classical computable model theory. This is simply because every Kripke model can be embedded into a classical model in a certain natural way [6]. Hence any construction of a Kripke model can be considered as a construction of a classical model. However, effectiveness considerations in intuitionistic model theory show that we can look at computable model theory with a different eye. This can be seen when one tries to compare effective versions of completeness results in classical model theory and in intuitionistic model theory. In this paper we give an example which shows that the effectiveness of the completeness result in classical model theory can sharply contrast with the effectiveness of completeness results for some nonclassical logics. The effective version of the completeness result for the classical predicate logic states that any decidable theory has a model whose full diagram is decidable. Such classical models are called **decidable**. In this paper we show that the completeness result, applied to a decidable theory over the logic with the weak law of excluded middle, produces a Kripke model for which the forcing (see Definition 1) is decidable in ω-jump of $\mathbf{0}$. We do not know if the ω–jump is the sharpest bound but we suspect that it is indeed so.

Here we mention some previous results concerning the effectiveness of the completeness theorem in intuitionistic model theory. Gabbay in [3] proved that for any decidable *finitely axiomatized* intuitionistic theory Γ and any sentence ϕ not intuitionistically derivable from Γ, there is a Kripke model of Γ which does not force ϕ, such that the underlying partially ordered set is a computable enumerable partial ordering, and such that forcing restricted to *atomic statements*

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is computably enumerable. In [7] a more sophisticated argument proves that any decidable intuitionistic theory Γ has a Kripke model $\mathcal{M}$ with decidable forcing such that for all sentences ϕ, ϕ is an intuitionistic consequence of Γ if and only if $\mathcal{M}$ forces ϕ. The proof in [7] guarantees only that the underlying partial ordering is a Π_2^0–set. However [8] shows that the underlying partial ordering can in fact be made computable.

We refer the reader to [8] for an elementary introduction to computable intuitionistic model theory with an emphasis to the effectiveness issue of completeness results.

2 Basic Notions

Kripke Frames and Models. Let $L =< P_0^{n_0}, \ldots, P_k^{n_k} \ldots, c_0, c_1, \ldots >$ be a computable language without function symbols. We denote the set of all sentences of L by $Sn(L)$.

A **frame** is a triple $F = (W, \leq, D)$ consisting of a non-empty set W, (**"states of knowledge"**), a partial order $\leq$ on W, and a map D from W to a power set such that $v \leq w$ implies $D(v) \subseteq D(w)$. D is called the **domain function**. The partially ordered set $(W, \leq)$ is called the **base** of the frame.

We suppose that we are given a mapping V, called a **valuation,** which assigns to each pair consisting of a $w \in W$ and an n–ary predicate symbol P (constant c) from L, a n–ary relation on $D(w)$ (element of $D(w)$).

Let $L(w)$ be the extension of the language L obtained by adding to L a constant (name) c_a for each element $a \in D(w)$. Let $A(w)$ be the set of all atomic sentences of language $L(w)$ classically true in $D(w)$ under the valuation V. Suppose that for all $v \leq w$ the set of all atomic sentences from $A(v)$ is a subset of $A(w)$. Then the 4–tuple $\mathcal{M} = (W, \leq, D, V)$ is called a **Kripke model (over frame F)**.

Definition 1. Let $(W, \leq, D, V)$ be a Kripke model of language L, w be in W and ϕ be a sentence from $L(w)$. We give the definition of "w **forces** ϕ" by induction on the complexity of ϕ.

1. For atomic sentences ϕ, w forces ϕ iff $\phi \in A(w)$.
2. w forces $\phi \to \psi$ iff for all $v \geq w$, v forces ϕ implies v forces ψ.
3. w forces $\neg\phi$ iff for all $v \geq w$, v does not force ϕ.
4. w forces $\forall x \phi$ iff for all $v \geq w$ and all constants $c \in L(v)$, v forces $\phi(c)$.
5. w forces $\exists x \phi$ iff for some $c \in L(w)$, w forces $\phi(c)$.
6. w forces $\phi \vee \psi$ iff w forces ϕ or w forces ψ.
7. w forces $\phi \& \psi$ iff w forces ϕ and w forces ψ.

$\mathcal{M}$ **forces** a ϕ if every $w \in W$ forces ϕ. One can prove that if w forces ϕ and $v \geq w$, then v forces ϕ.

Let Γ be a subset of $Sn(L)$. The **closure** of Γ is the set of all sentences which are intuitionistically deducible from Γ. A set Γ of sentences is **consistent** if the closure of Γ does not contain the falsehood $\perp$.

Computability Theory. We fix a standard effective enumeration $\Phi_0^X, \Phi_1^X, \ldots$ of all computable partial functions with oracle X. We call number n an *index* of

Φ_n^X. We assume that the reader knows basic facts about the arithmetical hierarchy, the jump operator and Turing degrees. $\mathbf{0}^n$ is the n–th jump of computable degree $\mathbf{0}$. $\mathbf{0}^\omega$ is the degree of the set $\{(x,i)|x \in \mathbf{0}^i\}$. The degree $\mathbf{0}^1$ is denoted by $\mathbf{0}'$. We refer to Soare [13] for the basic computability theory.

Intermediate Logics and Completeness. If we add the schema $\alpha \vee \neg\alpha$ to intuitionistic predicate logic **IPL**, then we obtain full classical predicate logic **CPL**. The logic **QJ** is obtained by adding the schema for the weak law of excluded middle

$$\neg\alpha \bigvee \neg\neg\alpha$$

to **IPL** and taking the closure. This logic is closed under substitution and intuitionistic deduction.

Definition 2. A logic S is **complete** for a class K of Kripke frames if the following two conditions hold:

1. All Kripke models over frames from K force all formulas from S.
2. For any $\alpha \in Sn(L)$ if α is not provable in S, then there is a Kripke model $\mathcal{M}$ over a Kripke frame in K such that $\mathcal{M}$ does not force α.

It is known that the classical predicate logic is complete for the class of antichain frames; The intuitionistic predicate logic **IPL** is complete for the class of tree frames; The logic **QJ** is complete for the class of directed frames; For proofs of these and other results and surveys of the subject, see [2] [6] [12] [4] [5] [8].

3 Extensions of Theories

We fix a language L and a logic S. When a sentence ϕ is intuitionistically deducible in logic S, we simply say that ϕ is deducible or S–deducible and write $\vdash_S \phi$.

Definition 3. 1. A **theory** T is a pair (Γ, Σ), where Γ and Σ are sets of sentences. We set $lT = \Gamma$ and $rT = \Sigma$.
2. A Kripke model $\mathcal{M}$ is **adequate** for T if for all sentences ϕ, ϕ is deducible from lT if and only if $\mathcal{M}$ forces ϕ.

$T = (\Gamma, \Sigma)$ is **inconsistent** if there exist $\alpha_1, \ldots, \alpha_n \in \Gamma$ and $\beta_1, \ldots, \beta_m \in \Sigma$ such that $\alpha_1 \& \ldots \& \alpha_n \to \beta_1 \vee \ldots \vee \beta_m$ is S–deducible. $T = (\Gamma, \Sigma)$ is **consistent** if it is not inconsistent.

Proposition 4. *Let $T = (\Gamma, \Sigma)$ be a consistent theory. Then there exists a theory $T = (\Gamma', \Sigma')$ such that (Γ', Σ') is consistent, $\Gamma' \bigcup \Sigma' = Sn(L)$, and $\Gamma \subset \Gamma'$ and $\Sigma \subset \Sigma$.*

Proof. Let $\alpha_0, \alpha_1, \ldots$ be a list of all sentences of the language L. We construct a sequence $(\Gamma_0, \Sigma_0), (\Gamma_1, \Sigma_1), \ldots$ of theories such that

1. For all $i \in \omega$, $\Gamma_i \subset \Gamma_{i+1}$, $\Sigma_i \subset \Sigma_{i+1}$,
2. For all $i \in \omega$, (Γ_i, Σ_i) is consistent,

3. $\bigcup_i(\Gamma_i \bigcup \Sigma_i) = Sn(L)$.

We build this sequence by stages.
Stage 0. Put $(\Gamma_0, \Sigma_0) = T_0 = (\Gamma, \Sigma)$.
Stage $n+1$. Take α_n. We have two cases.
Case 1. The theory $(\Gamma_n, \Sigma_n \bigcup \{\alpha_n\})$ is consistent. Then put $\Gamma_{n+1} = \Gamma_n$ and $\Sigma_{n+1} = \Sigma_n \bigcup \{\alpha_n\}$.
Case 2. $(\Gamma_n, \Sigma_n \bigcup \{\alpha_n\})$ is inconsistent. Then put $\Gamma_{n+1} = \Gamma_n \bigcup \{\alpha_n\}$ and $\Sigma_{n+1} = \Sigma_n$.
This ends the construction.
It is not hard to show that $T = (\Gamma', \Sigma')$ is the desired theory. □

Definition 5. A theory $T = (\Gamma, \Sigma)$ is **complete** if it is consistent and $Sn(L) = \Gamma \bigcup \Sigma$.

We also say that $T = (\Gamma', \Sigma')$ **extends** $T = (\Gamma, \Sigma)$ if $\Gamma \subset \Gamma'$ and $\Sigma \subset \Sigma'$. Thus, we have the following

Corollary 6. *Every consistent theory has a complete extension in the same language.* □

Definition 7. A proper subset Γ of $Sn(L)$ is **prime** if the following conditions are satisfied:

1. Γ is closed under deduction in S.
2. For all $\alpha, \beta \in Sn(L)$ if $\alpha \vee \beta \in \Gamma$, then either $\alpha \in \Gamma$ or $\beta \in \Gamma$.

For any subset $X \subset Sn(L)$ let $\bar{X}$ be the complement of X in $Sn(L)$, that is $\bar{X} = Sn(L) \setminus X$. It is not hard to see that the following proposition is true.

Proposition 8. *A set $\Gamma \subset Sn(L)$ is prime if and only if the theory $(\Gamma, \hat{\Gamma})$ is complete.* □

Definition 9. We say that a set Γ of sentences is Σ–**consistent** if $T = (\Gamma, \Sigma)$ is consistent. When $\Sigma = \{\beta\}$, then Σ–consistent set is called β–**consistent**.

Definition 10. A theory $T = (\Gamma, \Sigma)$ is **computable** if the deductive closure of Γ in logic S and the set Σ are computable.

Proposition 11. *Suppose that $T = (\Gamma, \Sigma)$ is a computable consistent theory and Σ is finite. Then T has a complete computable extension.*

Proof. Let $T = (\Gamma, \Sigma)$ be a computable consistent theory with Σ finite. Let $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ be a finite set of sentences. Then by the deduction theorem $\Gamma \bigcup \Delta$ proves ϕ if and only if Γ proves $\&_{i=1}^n \alpha_i \to \phi$. It follows that the closure of $\Gamma \bigcup \Delta$ is also computable. Therefore for finite subsets $\Delta_1, \Delta_2 \subset Sn(L)$, the theory $(\Gamma \bigcup \Delta_1, \Sigma \bigcup \Delta_2)$ is computable. Since Σ is finite and the closure of Γ is computable, the construction of the proof of Proposition 4 can be carried out effectively. The extension (Γ', Σ') obtained in the construction is a computable theory. □

Corollary 12. *Any consistent theory $T = (\Gamma, \Sigma)$ computable in X with Σ finite has a complete extension computable in X.* □

Let C be an infinite set of symbols, called **constants**, such that $L \bigcap C = \emptyset$. Put $L(C) = L \bigcup C$.

Definition 13. Let L be a language. A theory $T = (\Gamma, \Sigma)$ is **saturated** if $T = (\Gamma, \Sigma)$ is consistent, Γ is prime, and for every formula $\exists x \phi(x)$, the condition $\exists x \phi(x) \in \Gamma$ implies that $\phi(c) \in \Gamma$ for some constant c.

Proposition 14. *Every consistent theory $T = (\Gamma, \Sigma)$ of the language L can be extended to a saturated theory $T = (\Gamma', \Sigma')$ of the language $L(C)$.*

Proof. Let $\alpha_0, \alpha_1, \ldots$ be a list of all sentences of the language $L(C)$. We construct a sequence $(\Gamma_0, \Sigma_0), (\Gamma_1, \Sigma_1), \ldots$ of theories by stages.

Stage 0. Put $(\Gamma_0, \Sigma_0) = T = (\Gamma, \Sigma)$.

Stage $n+1$. Suppose that $T_n = (\Gamma_n, \Sigma_n)$ has been constructed. Take α_n. We have three cases.

Case 1. The theory $(\Gamma_n, \Sigma_n \bigcup \{\alpha_n\})$ is consistent. Then simply put $\Gamma_{n+1} = \Gamma_n$ and $\Sigma_{n+1} = \Sigma_n \bigcup \{\alpha_n\}$.

Case 2. The theory $(\Gamma_n, \Sigma_n \bigcup \{\alpha_n\})$ is inconsistent and α_n is not of the form $\exists x \beta(x)$. Then put $\Gamma_{n+1} = \Gamma_n \bigcup \{\alpha_n\}$ and $\Sigma_{n+1} = \Sigma_n$.

Case 3. The theory $(\Gamma_n, \Sigma_n \bigcup \{\alpha_n\})$ is inconsistent and α_n is of the form $\exists x \beta(x)$. Then put $\Gamma_{n+1} = \Gamma_n \bigcup \{\alpha_n, \beta(c)\}$ and $\Sigma_{n+1} = \Sigma_n$, where c is the first constant in C not used in the previous stages.

This ends the construction.

Put $\Gamma' = \bigcup_n \Gamma_n$ and $\Sigma' = \bigcup_n \Sigma_n$. Since at each stage $\alpha_n \in \Gamma_{n+1} \bigcup \Sigma_{n+1}$, we see that $Sn(L(C)) = \Gamma' \bigcup \Sigma'$. The theory $T' = (\Gamma', \Sigma')$ is consistent and saturated. □

Immediate corollaries are the following effective versions of the result above:

Proposition 15. *If $T = (\Gamma, \Sigma)$ is a computable consistent theory with finite Σ, then there exists a computable saturated extension $T' = (\Gamma', \Sigma')$ of $T = (\Gamma, \Sigma)$ in the expansion $L(C)$.* □

Corollary 16. *If $T = (\Gamma, \Sigma)$ is computable in X and is a consistent theory with finite Σ, then there exists a computable in X saturated extension $T' = (\Gamma', \Sigma')$ of $T = (\Gamma, \Sigma)$ in the expansion $L(C)$.* □

4 Decidable Kripke Models

We begin by defining the notion of decidable frame and Kripke model.

Definition 17. Let X be a set of natural numbers. A frame $(W, \leq, D)$ is **decidable in** X if the relation $w \in W \& w_1 \leq w_2 \& x \in D(w)$ is computable in X. If X is computable, then the frame is called **decidable**.

Definition 18. A Kripke model $(W, \leq, D, V)$ over a decidable X frame $(W, \leq, D)$ is X−**decidable** if the set

$$\{(w, \alpha(c_1, \ldots, c_n)) \mid w \in W, \alpha(c_1, \ldots, c_n) \in Sn(L(w)), w \text{ forces } \alpha(c_1, \ldots, c_n)\}$$

is computable in X. If X is computable, then the Kripke model is called **decidable**.

The below theorem is from [8] whose proof will be needed later.

Theorem 19. *Any computable theory $(\Gamma, \perp)$ has a decidable model $\mathcal{M}$ such that for all $\alpha \in Sn(L)$, α is deducible from Γ if and only if $\mathcal{M}$ forces α.*

Proof. We set $L_0 = L$ and $L_{n+1} = L(C_{n+1})$, where $C_1, C_2, \ldots$ is an effective sequence of infinite, uniformly computable, and pairwise disjoint sets of constant symbols. □

Lemma 20. *There exists an effective procedure p which for all $x, i \in \omega$ and all finite subsets Δ, if x is regarded as an index of a computable consistent theory (Γ, Δ) of the language L_i, produces an index $p(x, \Delta)$ of a computable complete saturated theory $(\Gamma(x, \Delta), \Sigma(x, \Delta))$ in the language L_{i+1} extending (Γ, Δ).*

Proof. The proof follows from the proof of Proposition 4. □

We want to define the base $(W, \leq)$ of the desired decidable adequate Kripke model for theory $(\Gamma, \perp)$.

Let $\alpha_0 \ldots \alpha_n$ be a sequence of sentences with the following properties:

1. Every α_i belongs to $Sn(L_i)$.
2. Every α_i is either of the form $\beta \to \gamma$ or $\forall y \beta(y)$.

We define a procedure described below which depends on $\alpha_0 \ldots \alpha_n$ and consists of at most $n+1$ steps.

Step 0. The step is **unsuccessful** if (Γ, α_0) is inconsistent. If this happens we terminate the procedure. Otherwise, we consider two cases:

Case 1. α_0 is of the form $\beta \to \gamma$. In this case $(\Gamma \bigcup \{\beta\}, \{\gamma\})$ is consistent. We effectively take an index x of this theory $(\Gamma \bigcup \{\beta\}, \{\gamma\})$. Applying Lemma 20, we get the theory $(\Gamma \bigcup \{\beta\}(x, \{\gamma\}), \Sigma(x, \{\gamma\}))$. We set $T(\alpha_0)$ to be $(\Gamma \bigcup \{\beta\}(x, \{\gamma\}), \Sigma(x, \{\gamma\}))$.

Case 2. α_0 is of the form $\forall y \beta(y)$. In this case there is a constant $c \in L_1$ such that $(\Gamma, \{\beta(c)\})$ is consistent. We effectively take an index x of this theory $(\Gamma, \{\beta(c)\})$. Applying Lemma 20, we get the theory $(\Gamma(x, \{\beta(c)\}), \Sigma(x, \{\beta(c)\}))$. We set $T(\alpha_0) = (\Gamma(x, \{\beta(c)\}),\ \Sigma(x, \{\beta(c)\}))$.

Step $i+1$, $i \leq n$. Suppose that $T(\alpha_0, \ldots, \alpha_i)$ has been constructed. Consider $lT(\alpha_1, \ldots, \alpha_i)$. The step is **unsuccessful** if $(lT(\alpha_1, \ldots, \alpha_i), \{\alpha_{i+1}\})$ is inconsistent. If this happens we terminate the procedure. Otherwise, consider two cases:

Case 1. α_{i+1} is $\beta \to \gamma$. In this case the theory $(lT(\alpha_1, \ldots, \alpha_i) \bigcup \{\beta\}, \{\gamma\})$ is consistent. We effectively take an index x of this theory. Applying Lemma 20, we get the theory $(lT(\alpha_1, \ldots, \alpha_i) \bigcup \{\beta\}(x, \{\gamma\}), \Sigma(x, \{\gamma\}))$. We set

$$T(\alpha_0, \ldots, \alpha_{i+1}) = (lT(\alpha_1, \ldots, \alpha_i) \bigcup \{\beta\}(x, \{\gamma\}, \Sigma(x, \{\gamma\})).$$

Case 2. α_{i+1} is of the form $\forall y\beta(y)$. In this case there is a constant $c \in L_{i+2}$ such that $(lT(\alpha_1,\ldots,\alpha_i),\{\beta(c)\})$ is consistent. We effectively compute an index x of this theory. Applying Lemma 20, we get the theory $(lT(\alpha_1,\ldots,\alpha_i)(x,\{\beta(c)\}), \Sigma(x,\{\beta(c)\}))$. We set

$$T(\alpha_1,\ldots,\alpha_{i+1}) = (lT(\alpha_1,\ldots,\alpha_i)(x,\{\beta(c)\}), \Sigma(x,\{\beta(c)\})).$$

This concludes the description of the procedure.

Definition 21. $\alpha_0\ldots\alpha_n$ is T**–ordered** if $T(\alpha_0,\ldots,\alpha_n)$ is defined.

Let W be the set of all T–ordered sequences. Let w, v be elements of W. We put $w \leq v$ if and only if w is an initial segment of v. The relation $\leq$ is computable and isomorphic to a disjoint union of countably many copies of an infinitely branching tree.

We define the frame $(W,\leq,D)$ as follows. Let $w = \alpha_0\ldots\alpha_n$. Then,

$$D(w) = \text{the set of all constants of the language } L_{n+1}.$$

This frame $(W,\leq,D)$ is computable. We define a valuation V on the frame as follows. Let $w = \alpha_0\ldots\alpha_n \in W$ and $P \in L$ be a predicate symbol. Then

$$P(c_1,\ldots,c_n) \text{ is (classically) true iff } P(c_1,\ldots,c_n) \text{ belongs to } lT(\alpha_0,\ldots,\alpha_n).$$

Thus, we have a Kripke model $(W,\leq D,V)$. Note that this model is decidable. A standard argument using induction on ϕ shows that a state of knowledge w forces a sentence ϕ if and only if ϕ belongs to $\Gamma(w)$. Moreover, for any $\phi \in Sn(L)$, ϕ is deducible from T if and only if ϕ is forced in model $\mathcal{M}$. The theorem is proved.

Corollary 22. *Any consistent theory $(\Gamma,\perp)$ computable in X has an X–decidable model $\mathcal{M}$ such that for all $\alpha \in Sn(L)$, α is deducible from Γ if and only if $\mathcal{M}$ forces α.*

Proof. Relativize the proof of the previous theorem. □

5 Computability of Adequate Models in QJ

We fix the logic **QJ** and begin with the investigation of computability of adequate models for computable theories over logic **QJ**. We follow ideas of the completeness proof of **QJ** from [4]. The completeness result for **QJ** states that **QJ** is complete for the class of directed Kripke frames. The goal of the section is to give an effective version of this completeness theorem. Our proof is an effectivization of the proof from [4]. We remind that a frame $F = (W,\leq,D)$ is **directed** if for all $v, w \in W$ there exists a $z \in W$ such that $v \leq z$ and $w \leq z$.

Theorem 23. *Let $T = (\Gamma,\perp)$ be a computable saturated theory over logic* **QJ**. *Then T possesses an adequate Kripke model which is decidable in $\mathbf{0}^\omega$ and whose base is a directed frame.*

Proof. We begin with considering the partially ordered set $(N^\star, \leq)$, where $N^\star$ is the set of all finite words over natural numbers, and $\leq$ is defined as follows. For $v, w \in N^\star$ $v \leq w$ iff w is an extension of v, that is, there exists a $z \in N^\star$ such that $v = wz$. λ denotes the empty word. Hence λ is the least element of $(N^\star, \leq)$. This partially ordered set is isomorphic to an infinitely branching tree. We fix a computable theory $T = (\Gamma, \perp)$ *with only one assumption,* that Γ is saturated. □

Definition 24. A **subordination model for** Γ is a triple $(N^\star, \leq, \hat{\Gamma})$ which satisfies the following properties.

1. $\hat{\Gamma}$ is a mapping which assigns to every $w \in N^\star$ a saturated theory $\hat{\Gamma}(w)$ of the language $L(w) = L + C(w)$, where $C(w)$ is an inifnite set of constants.
2. For all $v \leq w$, $L(v) \subset L(w)$ and $\hat{\Gamma}(w) \subset \hat{\Gamma}(w)$.
3. If $w_1 = wnv_1$, $w_2 = wkv_2$, and $n \neq k$, then $(C(w_1) \setminus C(w)) \bigcap (C(w_2) \setminus C(w)) = \emptyset$.
4. If $\alpha \to \beta \notin \hat{\Gamma}(w)$, then there exists an n such that $\alpha \in \hat{\Gamma}(wn)$ and $\beta \notin \hat{\Gamma}(wn)$.
5. If $\forall x \alpha(x) \notin \hat{\Gamma}(w)$, then there exists an n such that $\alpha(c) \notin \hat{\Gamma}(wn)$ for some $c \in C(wn)$.
6. $\hat{\Gamma}(\lambda) = \Gamma$.

Here is the lemma which shows that a subordination model for Γ carries all the information needed to construct a model of T. The proof of the lemma is standard.

Lemma 25. *Let $T = (\Gamma, \perp)$ be a saturated theory. Every subordination model $(N^\star, \leq, \hat{\Gamma})$ for Γ can be transformed into an adequate Kripke model $\mathcal{M}$ for T. Moreover the base of $\mathcal{M}$ is $(N^\star, \leq)$.* □

Definition 26. We say that a subordination model $(N^\star, \leq, \hat{\Gamma})$ for Γ is X–**decidable** if the set $\{(\phi, w) | \phi \in Sn(L_w) \& \phi \in \hat{\Gamma}(w)\}$ is computable in X. If X is a computable set, then the X–decidable subordination model is called **decidable**.

Lemma 27. *1. For every saturated theory $T = (\Gamma, \perp)$ computable in X, there exists an X–decidable subordination model for Γ.*

2. Every X–decidable subordination model $(N^\star, \leq, \hat{\Gamma})$ for Γ can be transformed into an X–decidable adequate Kripke model $\mathcal{M}$ for T. Moreover the base of $\mathcal{M}$ is $(N^\star, \leq)$.

Proof. Slightly modifying the proof of Theorem 19, one can see that every computable in X theory $T = (\Gamma, \perp)$ possesses an X–decidable subordination model for Γ[3]. The proof of the second part follows from the fact that if subordination model $(N^\star, \leq, \hat{\Gamma})$ is X–decidable, then the adequate Kripke model constructed in the previous lemma is X–decidable as well. □

[3] To see this, in the proof of Theorem 19 for every $w \in W$ and all immediate extensions $w\alpha$ of w introduce informally computable sequence $C_{w\alpha}$ of pairwise disjoint sets of new constants.

The next lemma, first proved in [4], uses the schema of the logic **QJ** and the definition of subordination model in an essential way.

Lemma 28. *Let $T = (\Gamma, \bot)$ be a saturated theory and let $(N^\star, \leq, \hat{\Gamma})$ be a subordination model for Γ. Then the set $\Gamma(\infty) = \bigcup_{w \in N^\star} \hat{\Gamma}(w)$ is $\bot$–consistent.*

Proof. It suffices to prove that for every $m \in \omega$, the set $\Gamma_m = \bigcup_{|w|=m} \hat{\Gamma}_w$ is $\bot$–consistent, where $|w|$ is the length of w. Suppose that there exists an m such that Γ_m is not $\bot$–consistent. We prove that in this case Γ_{m-1} is also not $\bot$–consistent.

Since Γ_m is not $\bot$–consistent there exist finite words $w_1k_1, \ldots, w_nk_n$ of length m such that $\hat{\Gamma}(w_1k_1) \bigcup \ldots \bigcup \hat{\Gamma}(w_nk_n)$ is not $\bot$–consistent. Hence, there exist sentences $\beta_1(\bar{a}_1, \bar{b}_1) \in \hat{\Gamma}(w_1k_1)$, …, $\beta_n(\bar{a}_n, \bar{b}_n) \in \hat{\Gamma}(w_nk_n)$ such that

$$\vdash_{\mathbf{QJ}} \beta_1(\bar{a}_1, \bar{b}_1) \& \ldots \& \beta_n(\bar{a}_n, \bar{b}_n) \to \bot,$$

where $b_i \in C(w_i)$, $a_i \in C(w_ik_i) \setminus C(w_i)$ for all i, $1 \leq i \leq n$. By the definition of subordination model we have $(C(w_ik_i) \setminus C(w_i)) \bigcap (C(w_jk_j) \setminus C(w_j)) = \emptyset$. Therefore, from intuitionistic logic we obtain

$$\vdash_{\mathbf{QJ}} \exists \bar{x}_1 \beta_1(\bar{x}_1, \bar{b}_1) \& \ldots \& \exists \bar{x}_n \beta_n(\bar{x}_n, \bar{b}_n) \to \bot\ .$$

Again from intuitionistic logic it also follows that

$$\vdash_{\mathbf{QJ}} \neg\neg\exists \bar{x}_1 \beta_1(\bar{x}_1, \bar{b}_1) \& \ldots \& \neg\neg\exists \bar{x}_n \beta_n(\bar{x}_n, \bar{b}_n) \to \bot\ .$$

Note that $\exists \bar{x}_i \beta_i(\bar{x}_i, \bar{b}_i) \in Sn(L(w_i))$. From the the fact that the logic is **QJ**, we see that

$$\hat{\Gamma}(w_i) \vdash_{\mathbf{QJ}} \neg\neg\exists \bar{x}_i \beta_i(\bar{x}_i, \bar{b}_i) \vee \neg\exists \bar{x}_i \beta_i(\bar{x}_i, \bar{b}_i).$$

Since $\hat{\Gamma}(w_i)$ is prime we get that $\neg\neg\exists \bar{x}_i \beta_i(\bar{x}_i, \bar{b}_i) \in \hat{\Gamma}(w_i)$ or $\neg\exists \bar{x}_i \beta_i(\bar{x}_i, \bar{b}_i) \in \hat{\Gamma}(w_i)$. It follows that $\neg\neg\exists \bar{x}_i \beta_i(\bar{x}_i, \bar{b}_i) \in \hat{\Gamma}(w_i)$. Consequently $\bigcup_{|w|=m-1} \hat{\Gamma}(w)$ is not $\bot$–consistent. This leads to a contradiction. □

Definition 29. Let $T = (\Gamma, \bot)$ be a theory. **An n–subordination model for** Γ is a triple $(\{0, \ldots, n\} \times N^\star, \leq, \hat{\Gamma})$ which satisfies the following properties.

1. $\hat{\Gamma}$ is a mapping which assigns to every $w \in \{0, 1, \ldots, n\} \times N^\star$ a saturated theory $\hat{\Gamma}_w$ of the language $L(w) = L + C(w)$.
2. For every $k \leq n$, the triple $(\{k\} \times N^\star, \leq^n, \hat{\Gamma}^n)$ is a subordination model for $\hat{\Gamma}((k, \lambda))$, where $\leq^k, \hat{\Gamma}^k$ are restrictions of $\leq, \hat{\Gamma}$ to $\{k\} \times N^\star$.
3. For all $k \leq n$, $\bigcup_{w<(k,\lambda)} \hat{\Gamma}(w) \subset \hat{\Gamma}((k, \lambda))$ and $\bigcup_{w<(k,\lambda)} C(w) \subset C((k, \lambda))$.
4. For all $k \leq n$, $(\hat{\Gamma}((k, \lambda)), \bot)$ is a saturated theory.
5. $\hat{\Gamma}((0, \lambda)) = \Gamma$.

Lemma 30. *Let $T = (\Gamma, \bot)$ be a theory. Every n–subordination model*

$$(\{0, 1, \ldots, n\} \times N^\star, \leq, \hat{\Gamma})$$

for Γ can be transformed into an adequate Kripke model $\mathcal{M}$ for T. Moreover the base of $\mathcal{M}$ is $(\{0, 1, \ldots, n\} \times N^\star, \leq)$. □

Theorem 31. *For every computable saturated theory $T = (\Gamma, \perp)$, there exists an adequate Kripke model $\mathcal{M}$ with the following properties:*

1. *The base of $\mathcal{M}$ is $(\{0, 1, \ldots, n\} \times N^\star, \leq)$.*
2. *The model $\mathcal{M}$ is decidable in $\mathbf{0}^n$.*

Proof. From Lemma 27, we see that every computable theory $T = (\Gamma, \perp)$ possesses a decidable subordination model $(N^\star, \leq, \bar{\Gamma})$ for Γ. Consider the theory $T = (\bigcup_{w \in N^\star} \hat{\Gamma}(w), \perp)$. This theory is computably enumerable. It follows that the deductive closure of $\bigcup_{w \in N^\star} \hat{\Gamma}(w)$ is computable in $\mathbf{0}'$. We can extend this theory to a saturated theory $T' = (\Gamma', \perp)$ over an expanded language such that T' is computable in $\mathbf{0}'$. Now we can develop a subordination model $(N^\star, \leq, \bar{\Gamma}')$ for Γ' in a such way that the set $\{(w, \phi) | \phi \in L_w \& \phi \in \bar{\Gamma}'(w)\}$ is computable in $\mathbf{0}'$. This shows that we can construct a 1–subordination model for Γ for which the set $\{(w, \phi) | \phi \in L(w) \& \phi \in \bar{\Gamma}'(w)\}$ is computable in $\mathbf{0}'$. Hence by the previous lemma we can transform this 1–subordination model into an adequate model of T which is decidable in $\mathbf{0}'$. Iterating this procedure $n-1$, we see that Γ has an n–subordination model $\{0, 1, \ldots, n\} \times N^\star, \leq, \bar{\Gamma}^{(n)}$ for which the set $\{(w, \phi) | \phi \in L(w) \& \phi \in \bar{\Gamma}^{(n)}(w)\}$ is computable in $\mathbf{0}^n$. □

Definition 32. Let $T = (\Gamma, \perp)$ be a saturated theory. An ω–**subordination model for** Γ is a triple $(\omega \times N^\star, \leq, \hat{\Gamma})$ such that.

1. $\hat{\Gamma}$ is a mapping which assigns to every $w \in \omega \times N^\star$ a saturated theory $\hat{\Gamma}_w$ of the language $L(w) = L + C(w)$.
2. For every $n \in \omega$, the triple $(\{n\} \times N^\star, \leq^n, \hat{\Gamma}^n)$ is a subordination model for $\hat{\Gamma}((n, \lambda))$, where $\leq^n, \hat{\Gamma}^n$ are restrictions of $\leq, \hat{\Gamma}$ to $\{n\} \times N^\star$.
3. For all $n \in \omega$, $\bigcup_{w < (n,\lambda)} \hat{\Gamma}(w) \subset \hat{\Gamma}((n, \lambda))$ and $\bigcup_{w < (n,\lambda)} C(w) \subset C((n, \lambda))$.
4. For all $n \in \omega$, $(\hat{\Gamma}((n, \lambda)), \perp)$ is a saturated theory.
5. $\hat{\Gamma}((0, \lambda)) = \Gamma$.

The following lemma is immediate

Lemma 33. *Let $T = (\Gamma, \perp)$ be a saturated theory. Every ω–subordination model $(\omega \times N^\star, \leq, \hat{\Gamma})$ for Γ can be transformed into an adequate Kripke model $\mathcal{M}$ for T. Moreover the base of $\mathcal{M}$ is $(\omega \times N^\star, \leq)$.* □

Proof of Theorem 23. Iterating the proof of Theorem 31 countably many times define a triple $(\omega \times N^\star, \leq, \hat{\Gamma})$ such that:

1. $(\{0\} \times N^\star, \leq^0, \hat{\Gamma}^0)$ is a decidable subordination model for Γ, where $\leq^0, \hat{\Gamma}^0$ are restrictions of $\leq$ and $\hat{\Gamma}$ to $\{0\} \times N^\star$
2. $\hat{\Gamma}(n, \lambda)$ is a saturated extension of $\bigcup_{w < (n,\lambda)} \hat{\Gamma}(w)$.
3. For every $n \in \omega$, $(\{0, 1, \ldots, n\} \times N^\star, \leq^n, \hat{\Gamma}^n)$ is a n–subordination model for Γ computable in $\mathbf{0}^n$, where $\leq^n$, $\hat{\Gamma}^n$ are restrictions of $\leq$ and $\hat{\Gamma}$ to $\{0, 1, \ldots, n\} \times N^\star$
4. The set $\{(\phi, w) | w \in \omega \times N^\star, \phi \in L(w), \phi \in \hat{\Gamma}(w)\}$ is computable in $\mathbf{0}^\omega$.

We see that the triple $(\omega \times N^{\star}, \leq, \hat{\Gamma})$ is an ω–subordination model for T. Hence this subordination model defines an adequate Kripke model $\mathcal{M}$ for T by Lemma 33. By the last item of the properties $\hat{\Gamma}$ we see that $\mathcal{M}$ is decidable in $\mathbf{0}^{\omega}$. □

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