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On algebraicness of D0L power series

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Abstract: We show that it is decidable whether or not a given D0L power series over a semiring A is A-algebraic in case $A = \mathbf{Q}_+$ or $A = \mathbf{N}$. The proof relies heavily on the use of elementary morphisms in a power series framework and gives also a new method to decide whether or not a given D0L language is context-free.

Category: F4.3

1 Introduction

D0L power series were defined in [Honkala 97] and studied in detail in [Honkala 98,00]. The study of these series gives an interesting counterpart to the customary theory of D0L systems.

In [Honkala 97] it is shown to be decidable whether or not a given D0L power series over $\mathbf{Q}$ is $\mathbf{Q}$-rational. In this paper we study the question whether or not a given D0L power series over a semiring A is A-algebraic. A decision method is provided in case A equals $\mathbf{Q}_+$ or $\mathbf{N}$. We also discuss the same question in case $A = \mathbf{Q}$. Our decision method relies heavily on the use of elementary morphisms in a power series framework and applies various techniques dealing with D0L sequences and algebraic series. By taking $A = \mathbf{B}$ we also obtain a new decision method for the context-freeness of D0L languages (see [Salomaa 75]).

For further background and motivation we refer to [Honkala 95,97,98,00] and the references given therein. It is assumed that the reader is familiar with the basics of formal power series and L systems (see [Berstel and Reutenauer 88], [Kuich and Salomaa 86], [Rozenberg and Salomaa 80,97], [Salomaa and Soittola 78]). Notions and notations that are not defined are taken from these references.

2 Definitions

Suppose A is a commutative semiring and X is a finite alphabet. The set of *formal power series with noncommuting variables* in X and coefficients in A is denoted by $A \ll X^* \gg$. The subset of $A \ll X^* \gg$ consisting of all series with a finite support is denoted by $A < X^* >$. Series of $A < X^* >$ are referred to as *polynomials*.

Assume that X and Y are finite alphabets. A semialgebra morphism $h : A < X^* > \longrightarrow A < Y^* >$ is called a *monomial morphism* if for each $x \in X$ there exist a nonzero $a \in A$ and $w \in Y^*$ such that $h(x) = aw$. If $h : A < X^* > \longrightarrow A < Y^* >$ is a monomial morphism, the *underlying monoid morphism* $\overline{h} : X^* \longrightarrow Y^*$ is defined by $\overline{h}(x) = \text{supp}(h(x))$ for $x \in X$. A series $r \in A \ll X^* \gg$ is called a D0L *power series* over A if there exist a nonzero $a \in A$, a word $w \in X^*$ and a monomial morphism $h : A < X^* > \longrightarrow A < X^* >$ such that

$$r = \sum_{n=0}^{\infty} ah^n(w) \tag{1}$$

and, furthermore,

$$\text{supp}(ah^i(w)) \neq \text{supp}(ah^j(w)) \text{ whenever } 0 \leq i < j.$$

Consider the series r given in (1) and denote

$$ah^n(w) = c_n w_n$$

where $c_n \in A$ and $w_n \in X^*$ for $n \geq 0$. Then we have

$$r = \sum_{n=0}^{\infty} c_n w_n. \tag{2}$$

In what follows the righthand side of (2) is called the *normal form* of r. A sequence $(c_n)_{n \geq 0}$ of elements of A is called a D0L *multiplicity sequence* over A if there exists a D0L power series r over A such that (2) is the normal form of r.

If $r = \sum_{n=0}^{\infty} ah^n(w)$ is a D0L power series and $p \geq 1$ and $m \geq 0$ are integers, then the series $r(p, m)$ is defined by

$$r(p, m) = \sum_{n=0}^{\infty} ah^{pn}(h^m(w)).$$

Assume that X and Y are finite alphabets. By definition, a monomial morphism $h : A < X^* > \longrightarrow A < Y^* >$ is *simplifiable* if there exist a set X_1 and monomial morphisms $h_1 : A < X^* > \longrightarrow A < X_1^* >$ and $h_2 : A < X_1^* > \longrightarrow A < Y^* >$ such that $h = h_2 h_1$ and $\text{card}(X_1) < \text{card}(X)$. If h is not simplifiable, it is called *elementary*. A D0L power series $r = \sum_{n=0}^{\infty} ah^n(w)$ is called *elementary* if the monomial morphism h is elementary.

3 Decidability of algebraicness in case $A = \mathbf{Q}_+$, $A = \mathbf{N}$ or $A = \mathbf{B}$

In this section we show through a sequence of lemmas that it is decidable whether or not a given D0L power series over the semiring A is A-algebraic in case $A = \mathbf{Q}_+$, $A = \mathbf{N}$ or $A = \mathbf{B}$. (Here $\mathbf{Q}_+$, $\mathbf{N}$ and $\mathbf{B}$ stand for the nonnegative rationals, nonnegative integers and Boolean semiring, respectively.) A decision method is first given for elementary D0L power series.

If X is a finite alphabet and $g : X^* \longrightarrow X^*$ is a morphism, a letter $x \in X$ is called *growing* if the set $\{g^n(x) \mid n \geq 0\}$ is infinite.

Lemma 1. *Suppose A is a commutative semiring and $r = \sum_{n=0}^{\infty} ah^n(w) \in A \ll X^* \gg$ is a D0L power series over A such that the underlying monoid morphism $g : X^* \longrightarrow X^*$ of h is injective. Furthermore, assume that there exist positive integers C and D such that*

$$|g^n(w)| \leq Cn + D$$

for all $n \geq 0$. Then there effectively exist integers $p \geq 1$, $q \geq 0$, $k \geq 0$, words $u_\alpha, v_\beta, w_\beta$ and growing letters y_β, $0 \leq \alpha \leq k$, $1 \leq \beta \leq k$, and nonzero $a_0, a_1, a_2 \in A$ such that

$$h^{np+q}(w) = a_0 a_1^n a_2^{\frac{(n-1)n}{2}} u_0(v_1^n y_1 w_1^n)u_1(v_2^n y_2 w_2^n)u_2 \ldots u_{k-1}(v_k^n y_k w_k^n)u_k \qquad (3)$$

for all $n \geq 0$. Furthermore, none of the words $u_\alpha, v_\beta, w_\beta$, $0 \leq \alpha \leq k$, $1 \leq \beta \leq k$, contains a growing letter.

Proof. Denote

$$X_1 = \{x \in X \mid |g^n(x)| = 1 \text{ for all } n \geq 1\}$$

and

$$X_2 = \{x \in X \mid x \text{ is a growing letter}\}.$$

If $x \in X_1$, clearly $g(x) \in X_1$. Hence g permutes the letters of X_1. If $x \in X$ is not growing, there exists a positive integer k such that $g^k(x) \in X_1^*$. Because g permutes the letters of X_1 there exists $u \in X_1^*$ such that $g^k(x) = g^k(u)$. Because g^k is injective, we have $x = u$ implying that $x \in X_1$. Consequently, $X = X_1 \cup X_2$.

Because $|g^n(w)|$ has a linear upper bound there exists a constant K such that no $g^n(w)$ contains more than K growing letters. Therefore there exist integers $p \geq 1$ and $q \geq 0$ such that

$$pr_{X_2}(g^q(w)) = pr_{X_2}(g^{p+q}(w)).$$

(Here pr_{X_2} is the projection from X^* onto X_2^*.) By changing p, if necessary, we may assume that $g^p(x) = x$ for all $x \in X_1$. Now, denote

$$h^q(w) = a_0 u_0 y_1 u_1 y_2 u_2 \ldots u_{k-1} y_k u_k \qquad (4)$$

where $k \geq 0$, $a_0 \in A$, $u_\alpha \in X_1^*$ and $y_\beta \in X_2$ for $0 \leq \alpha \leq k$, $1 \leq \beta \leq k$. Because each $g^p(y_\beta)$ contains only one growing letter, there exist $v_\beta, w_\beta \in X_1^*$ such that

$$g^p(y_\beta) = v_\beta y_\beta w_\beta$$

for $1 \leq \beta \leq k$. Finally, there exist nonzero $a_1, a_2 \in A$ such that

$$h^p(u_0 y_1 u_1 y_2 u_2 \ldots u_{k-1} y_k u_k) =$$
$$a_1 u_0(v_1 y_1 w_1)u_1(v_2 y_2 w_2)u_2 \ldots u_{k-1}(v_k y_k w_k)u_k$$

and

$$h^p(v_1 w_1 v_2 w_2 \ldots v_k w_k) = a_2 v_1 w_1 v_2 w_2 \ldots v_k w_k.$$

Now (3) follows inductively. First, if $n = 0$, (3) follows by (4). Then, if (3) holds, we have

$$h^{(n+1)p+q}(w) =$$
$$a_0 a_1^n a_2^{\frac{(n-1)n}{2}} h^p(u_0(v_1^n y_1 w_1^n)u_1(v_2^n y_2 w_2^n)u_2 \ldots u_{k-1}(v_k^n y_k w_k^n)u_k) =$$
$$a_0 a_1^{n+1} a_2^{\frac{n(n+1)}{2}} u_0(v_1^{n+1} y_1 w_1^{n+1})u_1(v_2^{n+1} y_2 w_2^{n+1})u_2 \ldots u_{k-1}(v_k^{n+1} y_k w_k^{n+1})u_k.$$

Hence (3) holds for all $n \geq 0$. □

Lemma 2. *Let $h : A < X^* > \longrightarrow A < X^* >$ be a monomial morphism such that there exist integers $p \geq 1$, $q \geq 0$, $k \geq 0$, words $u_\alpha, v_\beta, w_\beta$ and growing letters y_β, $0 \leq \alpha \leq k$, $1 \leq \beta \leq k$, and nonzero $a_0, a_1, a_2 \in A$ such that (3) holds for all $n \geq 0$ and none of the words $u_\alpha, v_\beta, w_\beta$, $0 \leq \alpha \leq k$, $1 \leq \beta \leq k$, contains a growing letter. Then there exist words $\overline{u}_\alpha, \overline{v}_\beta, \overline{w}_\beta, \overline{y}_\beta$, $0 \leq \alpha \leq k$, $1 \leq \beta \leq k$, such that*

$$h^{np+q}(w) = a_0 a_1^n a_2^{\frac{(n-1)n}{2}} \overline{u}_0 (\overline{v}_1^n \overline{y}_1 \overline{w}_1^n) \overline{u}_1 (\overline{v}_2^n \overline{y}_2 \overline{w}_2^n) \overline{u}_2 \ldots \overline{u}_{k-1} (\overline{v}_k^n \overline{y}_k \overline{w}_k^n) \overline{u}_k$$

for all $n \geq 0$. Furthermore, the following conditions hold: None of the words $\overline{u}_\alpha, \overline{v}_\beta, \overline{w}_\beta$ contains a growing letter. Each $\overline{y}_\beta$ contains exactly one growing letter. If $\overline{u}_\alpha = \lambda$ then either $\{\overline{w}_\alpha, \overline{v}_{\alpha+1}\}$ is a code or contains the empty word, $1 \leq \alpha \leq k-1$. If $\overline{u}_\alpha \neq \lambda$, then neither of the words $\overline{u}_\alpha$ and $\overline{w}_\alpha$ is a prefix of the other, $1 \leq \alpha \leq k-1$.

Proof. For each α, $1 \leq \alpha \leq k-1$, we modify the words $u_\alpha, v_\beta, w_\beta$ as follows. If $u_\alpha = \lambda$, $w_\alpha \neq \lambda$, $v_{\alpha+1} \neq \lambda$ and $\{w_\alpha, v_{\alpha+1}\}$ is not a code, replace w_α by $w_\alpha v_{\alpha+1}$, and $v_{\alpha+1}$ by λ, respectively. If $u_\alpha \neq \lambda$ and u_α is a prefix of w_α, replace y_α by $y_\alpha u_\alpha$, w_α by $u_\alpha^{-1} w_\alpha u_\alpha$, and u_α by λ, respectively. If $u_\alpha \neq \lambda$ and w_α is a prefix of u_α, replace y_α by $y_\alpha w_\alpha$, and u_α by $w_\alpha^{-1} u_\alpha$, respectively, and continue as before. When all these replacements are completed we have obtained the words $\overline{u}_\alpha, \overline{v}_\beta, \overline{w}_\beta, \overline{y}_\beta$, $0 \leq \alpha \leq k$, $1 \leq \beta \leq k$, satisfying the conditions of the claim. □

Lemma 3. *Denote*

$$r = \sum_{n=1}^{\infty} a_0 a_1^n a_2^{\frac{(n-1)n}{2}} \overline{u}_0 (\overline{v}_1^n \overline{y}_1 \overline{w}_1^n) \overline{u}_1 (\overline{v}_2^n \overline{y}_2 \overline{w}_2^n) \overline{u}_2 \ldots \overline{u}_{k-1} (\overline{v}_k^n \overline{y}_k \overline{w}_k^n) \overline{u}_k,$$

where $a_0, a_1, a_2 \in A$ are nonzero and the words $\overline{u}_\alpha, \overline{v}_\beta, \overline{w}_\beta, \overline{y}_\beta$, $0 \leq \alpha \leq k$, $1 \leq \beta \leq k$, satisfy the conditions of Lemma 2. Let t be the number of the words $\overline{v}_\beta$, $\overline{w}_\beta$, $1 \leq \beta \leq k$, when empty words are deleted and each nonempty word is counted as many times as it occurs. Let $z_1, \ldots, z_t$ be new distinct letters and denote

$$r_1 = \sum_{n=1}^{\infty} a_0 a_1^n a_2^{\frac{(n-1)n}{2}} z_1^n z_2^n \ldots z_t^n.$$

Then r is A-algebraic if and only if r_1 is A-algebraic.

Proof. First, suppose that r is A-algebraic. By the conditions stated in Lemma 2, each word in the language

$$\overline{u}_0 (\overline{v}_1^* \overline{y}_1 \overline{w}_1^*) \overline{u}_1 (\overline{v}_2^* \overline{y}_2 \overline{w}_2^*) \overline{u}_2 \ldots \overline{u}_{k-1} (\overline{v}_k^* \overline{y}_k \overline{w}_k^*) \overline{u}_k$$

can be written uniquely in the form

$$\overline{u}_0 (\overline{v}_1^{j_1} \overline{y}_1 \overline{w}_1^{j_2}) \overline{u}_1 (\overline{v}_2^{j_3} \overline{y}_2 \overline{w}_2^{j_4}) \overline{u}_2 \ldots \overline{u}_{k-1} (\overline{v}_k^{j_{2k-1}} \overline{y}_k \overline{w}_k^{j_{2k}}) \overline{u}_k$$

where $j_\gamma \in \mathbf{N}$ for $1 \leq \gamma \leq 2k$, provided that possibly different powers of empty words are not regarded as different. Because A-algebraic series are closed under inverse morphisms and Hadamard products with A-rational series, we may assume that the nonempty $\overline{u}_\alpha, \overline{v}_\beta, \overline{w}_\beta, \overline{y}_\beta$ are in fact distinct letters, $0 \leq \alpha \leq k$,

$1 \leq \beta \leq k$. Finally, we erase the letters corresponding to nonempty words $\overline{u}_\alpha, \overline{y}_\beta$, $0 \leq \alpha \leq k$, $1 \leq \beta \leq k$. The resulting series is still A-algebraic because at most three consecutive letters are erased (see [Kuich and Salomaa 86]).

Suppose then that r_1 is A-algebraic. By applying the closure properties of A-algebraic series it follows easily that r is A-algebraic. □

The following two lemmas recall some basic properties of algebraic series.

Lemma 4. *Suppose $A = \mathbf{Q}$ or $A = \mathbf{B}$. Let z be a letter and*

$$r = \sum_{i=0}^{\infty} a_i z^{n_i}$$

where $a_i \neq 0$ for $i \geq 0$, be a power series in $A \ll z^* \gg$. If

$$\lim_{i \to \infty} \frac{n_i}{i} = \infty$$

then r is not A-algebraic.

Proof. For both cases see [Kuich and Salomaa 86]. □

If $p \geq 2$ is a prime, denote by ν_p the p-adic valuation over $\mathbf{Q}$.

Lemma 5. *Suppose $r \in \mathbf{Q} \ll X^* \gg$ is $\mathbf{Q}$-algebraic and $p \geq 2$ is a prime. Then there exists a positive integer C such that*

$$|\nu_p((r, w))| \leq C|w|$$

for any nonempty word $w \in supp(r)$.

Proof. By Theorem IV6.6 in [Salomaa and Soittola 78] there exists a nonzero integer d such that

$$\sum (r, w) d^{|w|} w \in \mathbf{Z}^{\mathrm{alg}} \ll X^* \gg .$$

Furthermore, there exists a positive integer M such that

$$|(r, w) d^{|w|}| \leq M^{|w|}$$

for any nonempty $w \in X^*$. Hence there exists a positive integer D such that

$$0 \leq \nu_p((r, w) d^{|w|}) \leq D|w|$$

for any nonempty $w \in \mathrm{supp}(r)$. Consequently

$$-\nu_p(d)|w| \leq \nu_p((r, w)) \leq D|w|$$

for any nonempty $w \in \mathrm{supp}(r)$. This implies the claim. □

The following lemma gives our main result in the case of elementary D0L power series.

Lemma 6. *Suppose the basic semiring A equals $\mathbf{Q}_+$, $\mathbf{N}$ or $\mathbf{B}$. Then it is decidable whether or not a given elementary D0L power series $r = \sum_{n=0}^{\infty} ah^n(w)$ over A is A-algebraic.*

Proof. Let p_1 be the smallest period of the ultimately periodic sequence $(\mathrm{Alph}(h^n(w)))_{n\geq 0}$ and let q_1 be a nonnegative integer such that

$$\mathrm{Alph}(h^n(w)) = \mathrm{Alph}(h^{n+p_1}(w))$$

for all $n \geq q_1$. Because A-algebraic series are closed with respect to Hadamard products with A-rational series, if r is A-algebraic, so is $r(p_1, q_1)$. On the other hand, if $r(p_1, q_1)$ is A-algebraic, so is r, because

$$r = \sum_{n=0}^{q_1-1} ah^n(w) + \sum_{i=q_1}^{q_1+p_1-1} r(p_1, i) = \sum_{n=0}^{q_1-1} ah^n(w) + \sum_{i=q_1}^{q_1+p_1-1} h^{i-q_1}(r(p_1, q_1))$$

and h is nonerasing. So, it remains to decide whether or not $r(p_1, q_1)$ is A-algebraic.

Because h is nonerasing, the underlying D0L length sequence of $r(p_1, q_1)$ is strictly increasing. Next, decide whether or not the underlying D0L length sequence of $r(p_1, q_1)$ is linear. If not, Lemma 4 implies that r is not A-algebraic. We continue with the assumption that this sequence is linear. Then, by Lemma 1, there effectively exist integers $p \geq 1$, $q \geq 0$, $k \geq 0$, words $u_\alpha, v_\beta, w_\beta$ and growing letters y_β, $0 \leq \alpha \leq k$, $1 \leq \beta \leq k$, and nonzero $a_0, a_1, a_2 \in A$ such that

$$a(h^{p_1})^{np+q}(h^{q_1}(w)) =$$

$$a_0 a_1^n a_2^{\frac{(n-1)n}{2}} u_0(v_1^n y_1 w_1^n)u_1(v_2^n y_2 w_2^n)u_2 \ldots u_{k-1}(v_k^n y_k w_k^n)u_k$$

for all $n \geq 0$. Then we have

$$r(p_1, q_1)(p, q) = \sum_{n=0}^{\infty} a_0 a_1^n a_2^{\frac{(n-1)n}{2}} u_0(v_1^n y_1 w_1^n)u_1(v_2^n y_2 w_2^n)u_2 \ldots u_{k-1}(v_k^n y_k w_k^n)u_k.$$

Now, let L be the language of all words over the alphabet $\mathrm{Alph}(r(p_1, q_1))$ having length $|u_0 y_1 u_1 y_2 u_2 \ldots y_k u_k| + n|v_1 w_1 v_2 w_2 \ldots v_k w_k|$ for some $n \geq 0$. Because the underlying D0L length sequence of $r(p_1, q_1)$ is strictly increasing,

$$r(p_1, q_1) \odot \mathrm{char}(L) = r(p_1, q_1)(p, q).$$

(Here $s_1 \odot s_2$ stands for the Hadamard product of the series s_1 and s_2.) Hence, if $r(p_1, q_1)$ is A-algebraic, so is $r(p_1, q_1)(p, q)$. The converse is seen to be true as above.

Now, to decide whether or not $r(p_1, q_1)(p, q)$ is A-algebraic it suffices, by Lemmas 2 and 3 to decide whether or not the series

$$r_1 = \sum_{n=1}^{\infty} a_0 a_1^n a_2^{\frac{(n-1)n}{2}} z_1^n z_2^n \ldots z_t^n$$

is A-algebraic. Here t is an effectively obtainable integer and the letters z_γ are distinct. We claim that r_1 is A-algebraic if and only if $a_2 = 1$ and $t \leq 2$. First, if

r_1 is A-algebraic, Lemma 5 implies that $a_2 = 1$. Furthermore, if r_1 is A-algebraic, supp(r) is context-free. Consequently, $t \leq 2$. The converse implication follows immediately. □

In order to generalize Lemma 6 for arbitrary D0L power series a lemma is needed.

Lemma 7. *Let $h : A < X^* > \longrightarrow A < Y^* >$ be a monomial morphism. Then h is elementary if and only if the underlying monoid morphism $g : X^* \longrightarrow Y^*$ of h is elementary. If h is elementary, g is injective. If h is simplifiable, there exist a set X_1 and monomial morphisms $h_1 : A < X^* > \longrightarrow A < X_1^* >$ and $h_2 : A < X_1^* > \longrightarrow A < Y^* >$ such that $h = h_2 h_1$, card$(X_1) <$ card(X) and $h_2(x_1) \in Y^*$ for all $x_1 \in X_1$. Furthermore, the underlying monoid morphism $g_2 : X_1^* \longrightarrow Y^*$ of h_2 is injective.*

Proof. For the first claim see [Honkala 98]. The second claim follows by the first claim. Suppose then that h is simplifiable. If $h(x) \in A$ for all $x \in X$ the claim holds trivially. Otherwise, there exist a nonempty set X_1 and monoid morphisms $g_1 : X^* \longrightarrow X_1^*$, $g_2 : X_1^* \longrightarrow Y^*$ such that $g = g_2 g_1$ and card$(X_1) <$ card(X). By choosing as small X_1 as possible we may assume that g_2 is elementary. Now, denote $h(x) = a_x g(x)$ where $x \in X$ and $a_x \in A$, and define the monomial morphisms $h_1 : A < X^* > \longrightarrow A < X_1^* >$ and $h_2 : A < X_1^* > \longrightarrow A < Y^* >$ by

$$h_1(x) = a_x g_1(x), \qquad x \in X,$$

$$h_2(x) = g_2(x), \qquad x \in X_1.$$

Then, if $x \in X$ we have

$$h_2 h_1(x) = h_2(a_x g_1(x)) = a_x g_2 g_1(x) = a_x g(x) = h(x).$$

Furthermore, the underlying monoid morphism g_2 of h_2 is injective. □

Now we are ready for the main result.

Theorem 8. *Suppose the basic semiring A equals $\mathbf{Q}_+$, $\mathbf{N}$ or $\mathbf{B}$. Then it is decidable whether or not a given D0L power series $r = \sum_{n=0}^{\infty} a h^n(w)$ over A is A-algebraic.*

Proof. If h is elementary, apply the method of Lemma 6. If h is simplifiable, let h_1 and h_2 be as in Lemma 7 (where now $Y = X$.) Denote

$$r_1 = \sum_{n=0}^{\infty} a(h_1 h_2)^n (h_1(w)).$$

Hence, $r_1 \in A \ll X_1^* \gg$ is a D0L power series and

$$r = aw + h_2(r_1).$$

Because h_2 is nonerasing, the A-algebraicness of r_1 implies that of r. Conversely, if r is A-algebraic, so is r_1 because

$$g_2^{-1}\left(\sum_{u \neq w} (r, u) u\right) = r_1.$$

Consequently, it suffices to decide whether or not r_1 is A-algebraic. Continuing in the same way it is seen that after finitely many steps we are in a position to apply the method of Lemma 6. □

If the basic semiring A equals the Boolean semiring, Theorem 8 implies a new method to decide whether or not a given D0L language is context-free (see [Salomaa 75]).

4 The case $A = \mathbf{Q}$

In this section we briefly discuss the case $A = \mathbf{Q}$. We start with a problem concerning algebraic series.

Fix a semiring A. Let $X = \{x_i \mid i \in \mathbf{N}\}$ be an infinite alphabet and denote $X_k = \{x_1, x_2, \ldots, x_k\}$ for $k \geq 1$. Define the series $P_k \in A \ll X_k^* \gg$ by

$$P_k = \sum_{n=1}^{\infty} x_1^n x_2^n x_3^n \ldots x_k^n.$$

We claim that if P_{k+1} is A-algebraic, so is P_k, $k \geq 1$. For the proof, define the morphisms $g : X_{k+1}^* \longrightarrow X_k^*$ and $h : X_k^* \longrightarrow X_k^*$ by

$$g(x_i) = x_i^2 \quad \text{for} \quad 1 \leq i \leq k-1,$$

$$g(x_k) = g(x_{k+1}) = x_k$$

and

$$h(x_i) = x_i^2 \quad \text{for} \quad 1 \leq i \leq k.$$

Then we have $P_k = h^{-1}(g(P_{k+1}))$ which implies the claim by the closure properties of A-algebraic series.

Now, an integer k is called the *ALG-bound* for A if k is the largest integer such that P_k is A-algebraic. If no such k exists, the ALG-bound for A equals ∞. By the claim established above, P_k is A-algebraic if and only if k is at most the ALG-bound for A.

If A is a positive semiring the ALG-bound for A equals two. We do not know the ALG-bound for $A = \mathbf{Q}$.

Next, suppose the basic semiring A equals $\mathbf{Q}$. By the previous section it is decidable whether or not a given D0L power series over $\mathbf{Q}$ is $\mathbf{Q}$-algebraic. However, an explicit algorithm is obtained only if the ALG-bound for $\mathbf{Q}$ is known.

The decidability of algebraicness of D0L power series and the determination of ALG-bounds are closely related. In fact, if A is any semiring such that A-algebraicness is decidable for D0L power series over A then the ALG-bound for A is effectively computable if it is finite. This follows because P_k is A-algebraic if and only if the series

$$T_k = \sum_{n=1}^{\infty} y_1 x_1^n y_2 x_2^n \ldots y_k x_k^n$$

is A-algebraic. (Here $y_1, \ldots, y_k$ are new letters.) Furthermore, T_k is a D0L power series over A.

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