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A Note on Linear-Nondeterminism, Linear-Sized, Karp-Lipton Advice for the P-Selective Sets

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Abstract: Hemaspaandra and Torenvliet showed that each P-selective set can be accepted by a polynomial-time nondeterministic machine using linear advice and *quasi-linear* nondeterminism. We show that each P-selective set can be accepted by a polynomial-time nondeterministic machine using linear advice and *linear* nondeterminism.

Key Words: computational complexity, P-selectivity
Category: F.1

1 Introduction

The P-selective sets, sometimes referred to as the semi-feasible sets, were introduced by Selman [Sel79] as polynomial-time analogs of the semi-recursive sets of recursive function theory. They have played an active role in many facets of complexity theory (see [HNOS96b] and the survey [DHHT94] for references and discussion). By definition, the P-selective sets are those sets that have a polynomial-time function that chooses one of any two given strings, and that never chooses one that is out of the set if the other is in the set. (Informally, the function chooses one that is "no less likely than the other to be in the set.")

Definition 1 [Sel79]. A set $A \subseteq \Sigma^*$ is said to be *P-selective* if there is a (total) polynomial-time computable function $f : \Sigma^* \times \Sigma^* \rightarrow \Sigma^*$ (called a *P-selector function*) such that, for each x and y from Σ^*, it holds that

1. $f(x,y) \in \{x,y\}$, and
2. $\{x,y\} \cap A \neq \emptyset \Rightarrow f(x,y) \in A$.

P-sel will denote the class of all P-selective sets.

Karp and Lipton [KL80] introduced the study of nonuniform classes (advice classes). Their notion counts bits of advice, and thus is not the most precise method possible of counting information. For example, a string of (exactly) 3 bits has 8 possibilities, and one of 4 bits has 16 possibilities. If a given advice ranges over, say, 11 possibilities, we would like to be able to precisely capture that fact. To do this, we adopt the token-based advice refinement of Hemaspaandra and Torenvliet [HT96], and the following definitions follow exactly the presentation of that paper. As usual, $\mathbf{N}$ denotes $\{0,1,2,\ldots\}$ and $\mathbf{N}^+$ denotes $\{1,2,3,\ldots\}$.

Definition 2 [KL80]. 1. Let f be a function from $\mathbf{N}$ to $\mathbf{N}$. Let $\mathcal{C}$ be any collection of subsets of $\{0,1\}^*$. Define $\mathcal{C}/f(n) = \{A \mid (\exists B \in \mathcal{C})\, (\exists h : \mathbf{N} \rightarrow \{0,1\}^*)\, [(\forall n)\, [f(n) = |h(n)|]$ and $(\forall x \in \{0,1\}^*)\, [x \in A \iff \langle x, h(|x|)\rangle \in B]]\}$.

2. Let $\mathcal{F}$ be any class of functions from $\mathbf{N}$ to $\mathbf{N}$. Define

$$\mathcal{C}/\mathcal{F} = \{A \mid (\exists f \in \mathcal{F})\,[A \in \mathcal{C}/f]\}.$$

Regarding the class $\mathcal{F}$ above, we will use quadratic to denote the set of all quadratically bounded functions, and similarly for linear and quasilinear (by which we mean functions f such that $f(n) = \mathcal{O}(n \log^k(n))$ for some integer k). We will, throughout, follow the convention that / binds more tightly than $\cap$. For example, $\mathcal{C}/\mathrm{lin} \cap \mathcal{D}/\mathrm{lin}$ denotes $(\mathcal{C}/\mathrm{lin}) \cap (\mathcal{D}/\mathrm{lin})$, rather than $(\mathcal{C}/\mathrm{lin} \cap \mathcal{D})/\mathrm{lin}$.

Definition 3 (Token-based advice [HT96]). Let g be a function from $\mathbf{N}$ to $\mathbf{N}^+$. Assume natural numbers have their standard encoding over binary strings. Let $\mathcal{C}$ be any collection of subsets of $\{0,1\}^*$. Define

$$\mathcal{C}/\{g(n)\} = \{A \mid (\exists B \in \mathcal{C})(\exists h : \mathbf{N} \to \mathbf{N}^+)$$
$$[(\forall n)[h(n) \in \{1, \cdots, g(n)\}] \text{ and } (\forall x \in \{0,1\}^*)[x \in A \iff \langle x, h(|x|)\rangle \in B]]\}.$$

The relationship between the two notions is given by the following lemma.

Lemma 4 [HT96]. *For any class $\mathcal{C}$ closed under composition with logspace functions, and for any $f(n) : \mathbf{N} \to \mathbf{N}$,*

$$\mathcal{C}/\{2^{f(n)}\} = \mathcal{C}/f(n).$$

Ko proved that P-sel $\subseteq$ P/quadratic. This left open the issue of whether P-selective sets could be accepted with less than quadratic advice. Hemaspaandra, Naik, Ogihara, and Selman (implicit in [HNOS96a], see the discussion in [HT96]) resolved this in the affirmative, but to do so they had to enrich the power of the advice interpreters from deterministic to probabilistic: P-sel $\subseteq$ PP/lin. Hemaspaandra and Torenvliet [HT96] extended their result by reducing the power of the interpreter from PP to NP (note: PP $\supseteq$ NP):

$$\text{P-sel} \subseteq \text{NP/lin} \cap \text{coNP/lin}.$$

In fact, Hemaspaandra and Torenvliet pinpointed the exact amount of advice needed.

Theorem 5 [HT96]. *1.* P-sel $\subseteq$ NP/$\{2^n+1\}$ $\cap$ coNP/$\{2^n+1\}$.
2. P-sel $\not\subseteq$ NP/$\{2^n\}$ $\cup$ coNP/$\{2^n\}$.

Regarding the first part of this theorem, they showed something even stronger. They proved that quasilinear nondeterminism suffices for the NP linear-advice interpretation of P-selective sets. That is:

Theorem 6 [HT96]. P-sel $\subseteq$ TIME-NONDET[poly, $n(1+\log^* n)$]/$\{2^n+1\}$ $\cap$ (co-TIME-NONDET[poly, $n(1+\log^* n)$])/$\{2^n+1\}$.

In this paper, we note that the quasilinear nondeterminism can be reduced to linear nondeterminism: P-sel $\subseteq$

$$\text{TIME-NONDET[poly}, n]/\{2^n+1\} \cap (\text{co-TIME-NONDET[poly}, n])/\{2^n+1\},$$

and thus certainly

$$\text{P-sel} \subseteq \text{TIME-NONDET[poly, lin]/lin} \cap (\text{co-TIME-NONDET[poly, lin]})/\text{lin}.$$

We do so essentially by adopting the proof framework of Hemaspaandra and Torenvliet, augmented by a combinatorial fact that has been in the literature for decades. Richard Beigel (personal communication) has informed us that he has independently obtained essentially the same result.

2 Linear-Nondeterminism Interpretation

Theorem 7. P-sel $\subseteq$

$$\text{TIME-NONDET}[\text{poly}, n]/\{2^n + 1\} \cap (\text{co-TIME-NONDET}[\text{poly}, n])/\{2^n + 1\}.$$

Proof A tournament is a clique in which each edge is directed. Hohn, Landau, and Vaughan proved that in any n-node tournament there will be a node v such that every node in the tournament can be reached from v via a path of length at most two (see [Wes96] and the discussion in [Lan53]). So that our paper is self-contained, we include a proof of this result. It clearly holds for $n \leq 2$. Let T be an m-node tournament, $m > 2$. Let a be a node of T. Let w be a node of $T - \{a\}$ that by inductive hypothesis reaches each node of $T - \{a\}$ via a path of length at most two. If the edge between w and a points towards a we are done. So henceforward suppose the edge points towards w. If any node of T that is pointed to by w points to a we also are done. The only remaining case is that a points to w, and each node of T that is pointed to by w is also pointed to by a. However, in this case all of T can be reached from a via paths of length at most two (in particular, those nodes of $T - \{a\}$ that were reachable from w via paths of length 2 but not via any path of length 1 will be reachable from a via an edge from a to whatever node of $T - \{a\}$ the former length-2 path from w had as its intermediate node).

Let us be given an arbitrary P-selective set A. Let g be a polynomial-time computable P-selector function for A in the sense of Definition 1. Define $f(x, y) = g(\min(x, y), \max(x, y))$. Note that f is polynomial-time computable, is a P-selector function for A in the sense of Definition 1, and satisfies $(\forall x, y \in \Sigma^*)[f(x, y) = f(y, x)]$, that is, it is a symmetric function. We now define the advice for A at length n. If A has no length n strings let our advice be the token $2^n + 1$. Otherwise, let the length n strings in A induce a tournament via each being a node of the tournament, and there being an edge from a to b $(a \neq b)$ exactly if $f(a, b) = b$. By the result mentioned above, there is a node z from which each node of the tournament can be reached via paths of length at most two. z is a length n string. Let our advice for length n be the ith token, where z is the lexicographically ith length n string. Our nondeterministic, polynomial-time, linear-nondeterminism algorithm does the following. On an input x, $|x| = n$, it looks at the advice string (for length n). If the advice string is the token $2^n + 1$ it rejects. Otherwise, if the advice string is q, it computes the lexicographically qth string of length n, call it y_n. Then it nondeterministically guesses each length n string, b, and accepts (on the current nondeterministically guessed path) if and only if $f(b, y_n) = b$ and $f(x, b) = x$. (Note that this procedure will correctly handle even the degenerate cases in which $x = y_n$, or in which $f(x, y_n) = x$.) Due to the length-2 property of the advice string, and the definition of P-selectivity and the constraints it puts on the behavior of P-selectors, this algorithm will indeed accept x, $|x| = n$, exactly if x is in A. In particular, if $|x| = n$ then if $x \in A$ this procedure will clearly accept, and if $x \notin A$ this procedure cannot accept, as if it did either $f(b, y_n)$ or $f(x, b)$ would violate the behavior required of P-selector functions, as f would be outputting a member of $\overline{A}$ even though one of its arguments was an element of A (see Definition 1). So, P-sel $\subseteq$ TIME-NONDET[poly, n]/$\{2^n + 1\}$. The other half of the theorem's claim follows from the fact that P-sel is closed under complementation, and the fact that, clearly, $(\text{co-TIME-NONDET}[\text{poly}, n])/\{2^n + 1\}$ and $\text{co-}(\text{TIME-NONDET}[\text{poly}, n]/\{2^n + 1\})$ are equal. □

Corollary 8.

$$\text{P-sel} \subseteq \text{TIME-NONDET}[\text{poly}, \text{lin}]/\text{lin} \cap (\text{co-TIME-NONDET}[\text{poly}, \text{lin}])/\text{lin}.$$

Since

$$(\text{co-TIME-NONDET}[\text{poly}, \text{lin}])/\text{lin} = \text{co-}(\text{TIME-NONDET}[\text{poly}, \text{lin}]/\text{lin}),$$

the corollary could, of course, have been equally well stated using either of these names of that class.

As is standard, let $\text{E} = \cup_{c>0} \text{DTIME}[2^{cn}]$. We now discuss the fact that P-sel ⊆ E/lin ([BL97], see also [Nic97]), a fact that seems neither stronger nor weaker than the claims P-sel ⊆ PP/lin and P-sel ⊆ NP/lin. However, recall that Hemaspaandra et al. [HNOS96a] implicitly proved P-sel ⊆ PP/lin. It has already been noted in the literature (namely, in the long footnote of [HT96]) that their result also yields linear advice that can be obtained in exponential (i.e., $2^{\mathcal{O}(n)}$) time relative to the P-selective set the advice is for. We note that, furthermore, their approach in fact yields advice that can be *used* via probabilistic machines using a linear number of probabilistic moves. That is, Hemaspaandra et al. actually implicitly prove: P-sel ⊆ TIME-PROBABILISM[poly, lin]/lin. Since TIME-PROBABILISM[poly, lin] ⊆ E, this implies that P-sel ⊆ E/lin, a result that as noted earlier is directly obtained in work of Burtschick and Lindner [BL97]. We also mention that the linear advice used in the present paper (e.g., Theorem 7) can clearly be obtained in EF^A, where A is the P-selective set the advice is for, and EF denotes the E-time computable functions. Thus, the NP/lin advice in this paper has the same "easy to obtain relative to A" property as the PP/lin advice of Hemaspaandra et al. Finally, Torenvliet (personal communication, September 1997) has pointed out to us that, since TIME-NONDET[poly, lin] ⊆ E, Corollary 8 implies a different alternate route by which the Burtschick-Lindner result, P-sel ⊆ E/lin, can be obtained.

An interesting basic issue remains open. Does it hold that P-sel ⊆ P/lin? Of course, disproving this would immediately imply P ≠ NP [HT96].

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