Journal of Universal Computer Science, vol. 6, no. 1 (2000), 97-104
submitted: 15/6/99, accepted: 2/11/99, appeared: 28/1/00 © Springer Pub. Co.

Monotone, Horn and Quadratic Pseudo-Boolean Functions[1]

Stephan Foldes
(RUTCOR, Rutgers University,
640, Bartholomew Road,
Piscataway, NJ 08854-8003, USA
Email: sfoldes@rutcor.rutgers.edu)

Peter L. Hammer
(RUTCOR, Rutgers University,
640, Bartholomew Road,
Piscataway, NJ 08854-8003, USA
Email: hammer@rutcor.rutgers.edu)

Dedicated to Sergiu Rudeanu
with respect and affection.

Abstract: A pseudo-Boolean function (pBf) is a mapping from $\{0,1\}^n$ to the real numbers. It is known that pseudo-Boolean functions have polynomial representations, and it was recently shown that they also have disjunctive normal forms (DNFs). In this paper we relate the DNF syntax of the classes of monotone, quadratic and Horn pBfs to their characteristic inequalities.

Key Words: Pseudo-Boolean functions, Set functions, Boolean functions, Truth functions, Horn functions, Binary optimization.

Category: F.4 - Mathematical Logic and Formal Languages, G.2 - Discrete Mathematics

1 Introduction

An *n-adic pseudo-Boolean function* (or a *pseudo-Boolean function of n variables,* or simply *pBf*) is a mapping $f : B^n \to \mathcal{R}$, where $B = \{0,1\}$, $\mathcal{R}$ is the set of real numbers, and n is a positive integer. Join (disjunction), meet (conjunction) and complementation (negation) in B^n will be denoted by $V \vee W$, $V \wedge W = VW$, and $\bar{V}$, respectively. The order relation $V \leq W$ in B^n is defined componentwise. The symbols $\vee$ and $\wedge$ will also denote the max and min operators in $\mathcal{R}$.

A pseudo-Boolean function f is *Boolean* if its range is contained in B. A necessary and sufficient condition for this is that $f^2 = f$.

Pseudo-Boolean functions are essentially equivalent with set functions, i.e. mappings of the subsets of a finite set into the real field. The term pseudo-Boolean function reflects the similarity of these functions with the Boolean ones and was introduced in [Hammer, Rosenberg and Rudeanu 1963], while the class was amply studied from this perspective in [Hammer, Rudeanu 1968].

[1] C. S. Calude and G. Ştefanescu (eds.). *Automata, Logic, and Computability. Special issue dedicated to Professor Sergiu Rudeanu Festschrift.*

For every real number α, we have a particularly simple n-adic function, namely the constant function on B^n taking the value α.

The set of all n-adic functions constitutes a ring whose operations are defined by

$$(f+g)(V) = f(V) + g(V)$$
$$(fg)(V) = f(V) \times g(V)$$

On the same set a distributive lattice structure is defined as well, where the order and the join and meet operations are defined by

$$f \leq g \Leftrightarrow f(V) \leq g(V) \text{ for all } V \in B^n$$
$$(f \vee g)(V) = \max(f(V), g(V))$$
$$(f \wedge g)(V) = \min(f(V), g(V))$$

For any variable x with values in B, the function $1 \Leftrightarrow x$ is called the corresponding *complemented variable*, denoted also by $\bar{x}$. The functions x and $\bar{x}$ are called *Boolean literals*. Any function of the form $a + bx$, where a and b are constants and $b \neq 0$, is called a *pseudo-Boolean literal*. Every pseudo-Boolean literal has a unique expression $a + b\tilde{x}$, where $\tilde{x}$ is a Boolean literal (i.e. x or $\bar{x}$) and $b > 0$. Obviously a is the minimum value of such a literal, and $a + b$ is the maximum value of it. These concepts were developed in [Foldes, Hammer 2000].

An *elementary conjunction* was defined in [Foldes, Hammer 2000] as the greatest lower bound of one or more literals having the same minimum, i.e. a function of the form

$$(a + b_1\tilde{x}_1) \wedge \ldots \wedge (a + b_m\tilde{x}_m)$$

which can easily be shown to be equal to

$$a + (\min_i b_i)\tilde{x}_1 \ldots \tilde{x}_m$$

i.e. to

$$a + b\tilde{x}_1 \ldots \tilde{x}_m \tag{1}$$

where b and the b_i's are positive and each $\tilde{x}_i$ is a variable or a complemented variable.

An *elementary disjunction* was defined as the least upper bound of one or more literals having the same maximum; these are precisely the functions of the form

$$a + b\,(\tilde{x}_1 \vee \ldots \vee \tilde{x}_m) \tag{2}$$

Obviously, the minimum value of the elementary conjunction (1), as well as that of the elementary disjunction (2) is a, while $a + b$ is their maximum value (except in the case when (1) represents a constant elementary conjunction). Observe also that a non-constant elementary conjunction (1) or disjunction (2) is Boolean if and only if $a = 0$ and $b = 1$.

Let f be a pseudo-Boolean function. Any elementary conjunction g such that $g \leq f$ is called an *implicant* of f, and any maximal implicant of f is called a *prime* implicant. Any elementary disjunction h such that $f \leq h$ is called an *implicatum* of f, and any minimal implicatum is called a *prime* implicatum.

It was seen in [Foldes, Hammer 2000] that every pseudo-Boolean function f can be expressed as a finite join (disjunction) of elementary conjunctions having the same minimum a,

$$f = \bigvee_i \left[a + b_i(\tilde{x}_{i_1} \wedge \ldots \wedge \tilde{x}_{i_{m_i}})\right] \tag{3}$$

Such an expression is called a *disjunctive normal form* (DNF) representation of f. In particular, f has a DNF representation (3) in which the terms of the disjunction are all the different prime implicants of f. This is referred to as the *canonical* DNF of f.

Similarly, every pseudo-Boolean function f can be expressed as a finite meet (conjunction) of elementary disjunctions that have the same maximum t,

$$f = \bigwedge_i \left[a_i + b_i(\tilde{x}_{i_1} \vee \ldots \vee \tilde{x}_{i_{m_i}})\right] \tag{4}$$

where $a_i + b_i = t$ for every i. Such an expression is called a *conjunctive normal form* (CNF) representation of f. In particular f has a CNF representation (4) in which the terms of the conjunction are all the different prime implicata of f. This is referred to as the *canonical* CNF of f.

In this paper we shall examine several classes of pseudo-Boolean functions distinguished by particular forms of DNF representation, and we shall generalize to the case of pseudo-Boolean functions several characterizations given in [Ekin, Hammer and Peled 1997] for classes of Boolean functions.

This work was presented at the Meeting of the Institute for Operations Research and Management Science, Philadelphia, November 1999.

2 Monotone Pseudo-Boolean Functions

A pseudo-Boolean function f is called *monotone non-decreasing* if

$$V \leq W \text{ in } B^n \Longrightarrow f(V) \leq f(W) \text{ in } \mathcal{R}$$

and it is called *monotone non-increasing* if

$$V \leq W \text{ in } B^n \Longrightarrow f(V) \geq f(W) \text{ in } \mathcal{R}$$

For example, the function $\bar{x} \vee 2\bar{x}y \vee 2x \vee 4xy$ is monotone non-decreasing, while $y \vee 3\bar{y} \vee 2\bar{x}y$ is monotone non-increasing.

Observe that a literal $a + bx$ is monotone non-increasing or non-decreasing according to whether b is positive or negative.

The set of all monotone non-decreasing pseudo-Boolean functions is a sublattice of $\mathcal{R}^{B^n}$. Monotone non-increasing functions constitute another sublattice.

It is easy to see that a non-constant elementary conjunction (1) is monotone non-decreasing if and only if all Boolean literals occurring in (1) are variables, and it is monotone non-increasing if and only if all these literals are complemented variables. Similarly, a non-constant elementary disjunction (2) is monotone non-decreasing if and only if all the Boolean literals in (2) are variables, and it is monotone non-increasing if and only if all these literals are complemented variables.

The following two theorems generalize well known results from the theory of Boolean functions, which have also been extended recently to the class of discrete functions ([Bioch 1998]).

Theorem 1. *For any pseudo-Boolean function f the following conditions are equivalent:*

(i) *f is monotone non-decreasing,*
(ii) *some DNF of f contains no complemented variables,*
(ii') *the canonical DNF of f contains no complemented variables,*
(iii) *some CNF of f contains no complemented variables,*
(iii') *the canonical CNF of f contains no complemented variables.*

Proof: By the observations made before the statement of the Theorem, either of the properties (ii) or (iii) implies (i).

Let us show that (i) implies (ii'). Assume (i). Without loss of generality, let $a + b(\tilde{x}_1 \wedge \ldots \wedge \tilde{x}_m)$ be a prime implicant and assume $\tilde{x}_1 = x_1$. Suppose that one of the Boolean literals is a complemented variable, say $\tilde{x}_1 = 1 \Leftrightarrow x_1$. We shall derive a contradiction. Let $Q = \tilde{x}_2 \wedge \ldots \wedge \tilde{x}_m, P = \tilde{x}_1 \wedge \ldots \wedge \tilde{x}_m$. Because we have

$$\min\{f(V) : Q(V) = 1\} < \min\{f(V) : P(V) = 1\} = a + b$$

there must exist a $V = (v_1, \ldots, v_n)$ in B^n such that $Q(V) = 1,\ P(V) = 0$ and $f(V) < a + b$. Clearly, $v_1 = 1$. Let $W \in B^n$ be obtained from V by changing v_1 to 0. Then $P(W) = 1$ and therefore $a + b \leq f(W)$. Since $W < V$ in B^n, by (i) we must have $a + b \leq f(W) \leq f(V)$, contradicting $f(V) < a + b$. Thus (i) implies (ii') as claimed.

Let us also show that (i) also implies (iii'). Assume (i). Without loss of generality, let $a + b(\tilde{x}_1 \vee \ldots \vee \tilde{x}_m)$ be a prime implicatum. As in the preceding part, we shall derive a contradiction. Let $D = \tilde{x}_1 \vee \ldots \vee \tilde{x}_m$, $Q = \tilde{x}_2 \vee \ldots \vee \tilde{x}_m$. Because we have

$$\max\{f(V) : Q(V) = 0\} > \max\{f(V) : D(V) = 0\} = a$$

there is a $V = (v_1, \ldots, v_n)$ in B^n such that $Q(V) = 0,\ D(V) = 1$ and $f(V) > a$. We must have $v_1 = 0$. Let $W \in B^n$ be obtained from V by changing v_1 to 1. Then $D(W) = 0$ and therefore $f(W) \leq a$. Since $V < W$ in B^n, by (i) we must have $f(V) \leq f(W) \leq a$, contradicting $f(V) > a$. Thus (i) implies (iii').

Finally, (ii) and (iii) follow from (ii') and (iii') respectively, and we have already noted that each one of them implies (i). □

As an example, consider the monotone non-decreasing function $\bar{x} \vee 2\bar{x}y \vee 2x \vee 4xy$ on B^2. Its prime implicants are $(1 + x)$, $(1 + y)$ and $(1 + 3xy)$. The prime implicata are $(2 + 2x), (2 + 2y)$ and $1 + 3(x \vee y)$.

Theorem 2. *For any pseudo-Boolean function f the following conditions are equivalent:*

(i) *f is monotone non-increasing,*
(ii) *in some DNF of f, all variable occurrences are complemented,*
(ii') *all Boolean literals occurring in the canonical DNF of f are complemented variables,*

(iii) *in some CNF of f all variable occurrences are complemented,*
(iii') *all Boolean literals in the canonical CNF of f are complemented variables.*

Proof: Observe that

(A) f is monotone non-increasing if and only if $\Leftrightarrow f$ is monotone non-decreasing
(B) the prime implicants of f are in bijective correspondence with the prime implicata of $\Leftrightarrow f$, where for a prime implicant $a + b(\tilde{x}_1 \wedge \ldots \wedge \tilde{x}_m)$ of f the corresponding prime implicatum of $\Leftrightarrow f$ is

$$\Leftrightarrow a \Leftrightarrow b(\tilde{x}_1 \wedge \ldots \wedge \tilde{x}_m) = (\Leftrightarrow a \Leftrightarrow b) + b[(1 \Leftrightarrow \tilde{x}_1) \vee \ldots \vee (1 \Leftrightarrow \tilde{x}_m)]$$

Using these observations, Theorem 1 can be used to establish the equivalences of Theorem 2. □

As an example, consider the monotone non-increasing function $y \vee 3\bar{y} \vee 2\bar{x}y$ on B^2. Its prime implicants are $1 + \bar{x}$ and $1 + 2\bar{y}$. The prime implicata are $2 + \bar{y}$ and $1 + 2(\bar{x} \vee \bar{y})$.

3 Pseudo-Boolean Horn Functions

A Boolean function is called a *Horn function* if it has a DNF having at most one complemented variable in each of its terms, and such a DNF is called a *Horn DNF*. Adopting the same definitions for *pseudo-Boolean Horn functions* and *Horn DNF's*, the following generalization of a known characterization of Horn functions among Boolean functions (a Boolean function is Horn if and only if $f(VW) \leq f(V) \vee f(W)$ for any $V, W \in B^n$; see [Ekin, Hammer and Peled 1997]) holds:

Theorem 3. *A pseudo-Boolean function f has a Horn DNF if and only if $f(VW) \leq f(V) \vee f(W)$ for all $V, W \in B^n$.*

Proof: Suppose that all the elementary conjunctions $P_i = \tilde{x}_{i_1} \ldots \tilde{x}_{im_i}$ are Horn, but $f(VW) > f(V) \vee f(W)$ for some $V, W \in B^n$. Define

$$g = \vee\{P_i : a + b_i \geq f(VW)\}$$

Then g is a Horn Boolean function and

$$1 = g(VW) < g(V) \vee g(W) = 0$$

In view of the quoted above result of [Ekin, Hammer and Peled 1997], this is impossible, thus proving the validity of $f(VW) \leq f(V) \vee f(W)$.

Conversely, suppose $f(VW) \leq f(V) \vee f(W)$ valid. Suppose some prime implicant is of the form $a + b\bar{x}\bar{y}P$: we shall derive a contradiction, proving that the canonical DNF is of the required form. Since $a + b\bar{x}\bar{y}P$ is prime, neither $a + b\bar{x}P$ nor $a + b\bar{y}P$ are implicants, i.e. there exist $V, W \in B^n$ such that (denoting without loss of generality $x = x_1, y = x_2$)

$$a + b\bar{x}_1 P(V) = a + b,\ \ f(V) < a + b$$
$$a + b\bar{x}_2 P(W) = a + b,\ f(W) < a + b$$

We must have $b\bar{x}_1\bar{x}_2 P(V) = b\bar{x}_1\bar{x}_2 P(W) = 0$, and thus $v_1 = 0, v_2 = 1, w_1 = 1, w_2 = 0$, while $P(V) = P(W) = 1$. Therefore both the first and second components of VW are 0, and $P(VW) = 1$. It follows that $b\bar{x}_1\bar{x}_2 P(VW) = b$, implying $f(VW) \geq a + b$, hence $f(V) \geq a + b$ or $f(W) \geq a + b$, which is impossible. □

4 Quadratic Pseudo-Boolean Functions

A Boolean function is called *quadratic* if it is a constant or if it has a DNF of the form $\bigvee_i P_i$, where each P_i is the conjunction of at most two Boolean literals. It was shown in [Ekin, Hammer and Peled 1997] a Boolean function is quadratic if and only if it satisfies

$$f(UV \vee UW \vee VW) \leq f(U) \vee f(V) \vee f(W) \tag{5}$$

for any $U, V, W \in B^n$. We shall prove below the following generalization of this result:

Theorem 4. *A pseudo-Boolean function f has a DNF $\bigvee_i (a+b_i(\tilde{x}_{i1} \wedge \ldots \wedge \tilde{x}_{im_i}))$ with all Boolean conjunctions $P_i = \tilde{x}_{i1} \ldots \wedge \tilde{x}_{im_i}$ quadratic if and only if it satisfies*

$$f(UV \vee UW \vee VW) \leq f(U) \vee f(V) \vee f(W) \tag{6}$$

for all $U, V, W \in B^n$.

Proof: Suppose $f = \vee(a + b_i P_i)$ with every P_i quadratic. If we had for some $U, V, W \in B^n$,

$$f(UV \vee UW \vee VW) > f(U) \vee f(V) \vee f(W) \tag{7}$$

then the quadratic Boolean function g defined by

$$g = \bigvee \{P_i : a + b_i \geq f(UV \vee UW \vee VW)\}$$

would not satisfy (6) because we would have

$$1 = g(UV \vee UW \vee VW) > g(U) \vee g(V) \vee g(W) = 0$$

Therefore we cannot have (7), and hence (6) must be valid.

Conversely, suppose (6) is valid. It suffices to show that no prime implicant of f is of the form $a + b\tilde{x}_1\tilde{x}_2\tilde{x}_3P$, where b is positive, $\tilde{x}_i$ is either x_i or $\bar{x}_i$, and x_1, x_2, x_3 are three distinct variables. Assume that, to the contrary, we have such a prime implicant

$$h = a + b\tilde{x}_1\tilde{x}_2\tilde{x}_3P$$

Let $h_3 = a + b\tilde{x}_1\tilde{x}_2P$, $h_2 = a + b\tilde{x}_1\tilde{x}_3P$, $h_1 = a + b\tilde{x}_2\tilde{x}_3P$. From the primality of h it follows that none of the h_i is an implicant, and hence, that there are $U, V, W \in B^n$ such that

$$h_3(U) = h_2(V) = h_1(W) = a + b,$$

while $f(U), f(V), f(W)$ are all less than $a + b$. We must have

$$\begin{array}{c}
u_3 \neq v_3,\ v_3 = w_3,\ \tilde{x}_3(U) = 0, \tilde{x}_3(V) = \tilde{x}_3(W) = 1 \\
v_2 \neq u_2,\ u_2 = w_2,\ \tilde{x}_2(V) = 0, \tilde{x}_2(U) = \tilde{x}_2(W) = 1 \\
w_1 \neq u_1,\ u_1 = v_1,\ \tilde{x}_1(W) = 0,\ \tilde{x}_1(U) = \tilde{x}_1(V) = 1
\end{array}$$

It follows that

$$\tilde{x}_1(UV) = \tilde{x}_2(UW) = \tilde{x}_3(VW) = 1$$

and

$$x_2(UV) = x_3(UV) = 0$$
$$x_1(UW) = x_3(UW) = 0$$
$$x_1(VW) = x_2(VW) = 0$$

Consequently all the $\tilde{x}_i,\ i = 1, 2, 3$, take the value 1 on $UV \vee UW \vee VW$, and obviously so does P. But then

$$a + b = h(UV \vee UW \vee VW) \leq f(UV \vee UW \vee VW)$$

and hence by the validity of (6), at least one of $f(U), f(V), f(W)$ must be greater than or equal to $a + b$: contradiction. □

5 Applications

The *maximization* of a pBf in DNF representation is clearly achievable in polynomial time. On the other hand, the *minimization* problem is obviously intractable, since SAT is a particular case of it.

Some important practical pBf minimization problems concern functions given in DNF representation involving monotone, quadratic or Horn expressions (see e.g. [Boros, Hammer, Minoux and Rader 1999] for an application in VLSI design). The minimization of a monotone DNF is trivial. The minimization of a quadratic or Horn DNF can be achieved in polynomial time by a simple reduction to the Boolean case.

Indeed, it follows by distributivity from the DNF representation (3), that any pBf f can be written as

$$f = a + \left(\bigvee_{i=1}^{t} b_i \tilde{x}_{i_1} \ldots \tilde{x}_{i_{m_i}} \right)$$

where all $b_i > 0$. The way to check whether for a real number r there exists an $X \in B^n$ with $f(X) \leq r$, we have to check whether the system

$$\bigvee_{i \in A} \tilde{x}_{i_1} \ldots \tilde{x}_{i_{m_i}} = 1$$
$$\bigvee_{i \in B} \tilde{x}_{i_1} \ldots \tilde{x}_{i_{m_i}} = 0$$

is consistent; here $A = \{i : \ a + b_i \leq r\}$, $B = \{i : \ a + b_i > r\}$. It is well known that in case of quadratic or Horn DNFs the above Boolean equations' consistency can be established in polynomial time. Since the minimum of f can only occur in one of the values $a, a + b_1, \ldots, a + b_t$, the polynomiality of the minimization problem for quadratic and Horn DNFs follows immediately.

Acknowledgements

This work was partially supported by NSF grant DMS-9806389 and ONR grant N00014-92-J-1375.

References

[Bioch 1998] Bioch, J.C.: "Dualization, decision lists and identification of monotone discrete functions"; Annals of Mathematics and Artificial Intelligence, 24 (1998), 69-91.

[Boros, Hammer, Minoux and Rader 1999] Boros, E., Hammer, P.L., Minoux, M., Rader, D.: "Optimal cell flipping to minimize channel density in VLSI design and pseudo-Boolean optimization"; Discrete Applied Mathematics, 90 (1999), 69-88.

[Ekin, Hammer and Peled 1997] Ekin, O., Hammer, P.L., Peled, U.: "Horn functions and submodular Boolean functions"; Theoretical Computer Science, 175 (1997), 257-270.

[Foldes, Hammer 2000] Foldes, S., Hammer, P.L.: "Normal forms of pseudo-Boolean functions"; RRR 1-2000, Rutgers University, RUTCOR Research Reports (2000), http://rutcor.rutgers.edu/.

[Hammer, Rosenberg and Rudeanu 1963] Hammer, P.L., Rosenberg, I.G., Rudeanu, S.: "On the minimization of pseudo-Boolean functions"; Stud. Cerc. Matem. 14 (1963), 359-364.

[Hammer, Rudeanu 1968] Hammer, P.L., Rudeanu, S.: "Boolean Methods in Operations Research and Related Areas"; Springer (1968).