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A Representation Theorem for Monadic Pavelka Algebras[1]

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Abstract: In this paper we define the *monadic Pavelka algebras* as algebraic structures induced by the action of quantifiers in Rational Pavelka predicate logic. The main result is a representation theorem for these structures.

Key Words: Pavelka algebra, monadic Pavelka algebra, MV-algebra, monadic MV-algebra.
Category: F.4.1.

1 Introduction

Rational Pavelka logic (RPL) is obtained from Łukasiewicz infinite valued propositional calculus (Ł) by adding the truth constants $\overline{r}$ for $r \in [0,1] \cap \mathrm{Q}$. The corresponding algebraic structures (*Pavelka algebras*) will be MV-algebras that contain a set of constants $\{\overline{r} \mid r \in [0,1] \cap \mathrm{Q}\}$ as a subalgebra. The quantifiers defined on an MV-algebra appear in [10, 11] reflecting the action of the quantifiers in Łukasiewicz infinite valued predicate calculus (Ł$\forall$). In this paper we start from the Rational Pavelka predicate logic (RPL$\forall$) in order to define the quantifiers on Pavelka algebras. This leds to the notion of *monadic Pavelka algebra*. If K is a non-empty set then the MV-algebra $[0,1]^K$ has a canonical structure of monadic Pavelka algebra. The main result of this paper is a representation theorem for monadic Pavelka algebras. In fact, our results can be viewed as algebraic versions of the results in [6] (see also [4], pp. 223-226).

2 Monadic MV-algebras

The MV-algebras were introduced in [1] as algebraic models for Ł. An *MV-algebra* is an algebraic structure $\langle A, \oplus, \neg, 0\rangle$ where $\langle A, \oplus, 0\rangle$ is an abelian monoid and $\neg$ is an unary operation such that :
1. $\neg\neg x = x$ for any $x \in A$,
2. $x \oplus \neg 0 = \neg 0$ for any $x \in A$,
3. $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$ for any $x, y \in A$.

[1] C. S. Calude and G. Ştefănescu (eds.). *Automata, Logic, and Computability. Special issue dedicated to Professor Sergiu Rudeanu Festschrift.*

We also define $1 = \neg 0$, $x \odot y = \neg(\neg x \oplus \neg y)$, $x \longrightarrow y = \neg x \oplus y$, $x \vee y = x \oplus (\neg x \odot y)$, $x \wedge y = x \odot (\neg x \oplus y)$. Thus $\langle A, \vee, \wedge, 0, 1\rangle$ is a bounded distributive lattice. If $x \in A$ and n is a natural number we denote

$0x = 0$, $(n+1)x = nx \oplus x$,

$x^0 = 1$, $x^{n+1} = x^n \odot x$.

The interval $[0,1]$ is an MV-algebra with respect to the operations $x \oplus y = min(1, x+y)$ and $\neg x = 1 - x$. In $[0,1]$ we have that $x \odot y = max(0, x+y-1)$ and $x \longrightarrow y = min(1, 1-x+y)$. If $x < 1$ then there exists a natural number n such that $x^n = 0$.

Lemma 2.1 [2] *In every MV-algebra A the following equalities hold:*
(i) $a \oplus \bigvee_{i\in I} x_i = \bigvee_{i\in I}(a \oplus x_i)$, $a \oplus \bigwedge_{i\in I} x_i = \bigwedge_{i\in I}(a \oplus x_i)$,
(ii) $a \odot \bigvee_{i\in I} x_i = \bigvee_{i\in I}(a \odot x_i)$, $a \odot \bigwedge_{i\in I} x_i = \bigwedge_{i\in I}(a \odot x_i)$,
(iii) *if A is linearly ordered then*
$\bigvee_{i\in I}(x_i \oplus x_i) = 2\left(\bigvee_{i\in I} x_i\right)$, $\bigvee_{i\in I}(x_i \odot x_i) = \left(\bigvee_{i\in I} x_i\right)^2$,
$\bigwedge_{i\in I}(x_i \oplus x_i) = 2\left(\bigwedge_{i\in I} x_i\right)$, $\bigwedge_{i\in I}(x_i \odot x_i) = \left(\bigwedge_{i\in I} x_i\right)^2$.

Lemma 2.2 [1, 2] *The implication operation $\longrightarrow$ has the following properties:*
(i) $x \le y$ *iff* $x \longrightarrow y = 1$,
(ii) $y \odot z \le x$ *iff* $y \le z \longrightarrow x$,
(iii) $(x \vee y) \longrightarrow z = (x \longrightarrow z) \wedge (y \longrightarrow z)$.

A non-empty subset F of A is an *MV-filter* (*filter*) if for every x, $y \in A$ the following are satisfied:
4. $x,y \in F \Rightarrow x \odot y \in F$,
5. $x \le y$, $x \in F \Rightarrow y \in F$.
For $X \subseteq A$ the filter generated by X is given by

$filt(X) = \{a \in A \mid x_1 \oplus \cdots \oplus x_n \le a$ for some $n < \omega$ and $x_1, \ldots, x_n \in X\}$.

If F is a filter and $b \in A$ then

$filt(X \cup \{b\}) = \{a \in A \mid x \odot b^n \le a$ for some $n < \omega$ and $x \in F\}$.

With any filter F of A we can associate a congruence $\sim_F$ on A:

$x \sim_F y$ iff $(x \longrightarrow y) \wedge (y \longrightarrow x) \in F$.

Denote by $A/_F$ the quotient MV-algebra $A/\sim_F$ and denote by $a/_F$ the class of $a \in A$.

A proper filter P is *prime* if $x \vee y \in P$ implies $x \in P$ or $y \in P$. One can prove that a proper filter P is prime iff $x \longrightarrow y \in P$ or $y \longrightarrow x \in P$ for any x, $y \in A$ iff $A/_P$ is a linearly ordered MV-algebra.

Definition 2.3 *An* existential quantifier *on an MV-algebra A is a mapping $\exists : A \longrightarrow A$ which satisfies the following axioms:*
M0. $\exists 0 = 0$,
M1. $x \le \exists x$,
M2. $\exists(x \odot \exists y) = \exists x \odot \exists y$,
M3. $\exists(x \oplus \exists y) = \exists x \oplus \exists y$,
M4. $\exists(x \odot x) = \exists x \odot \exists x$,
M5. $\exists(x \oplus x) = \exists x \oplus \exists x$.

If we define $\forall x = \neg\exists\neg x$ for any $x \in A$ then the mapping $\forall : A \longrightarrow A$ fulfils the following properties:
M0°. $\forall 1 = 1$,

M1°. $\forall x \leq x$,
M2°. $\forall(x \oplus \forall y) = \forall x \oplus \forall y$,
M3°. $\forall(x \odot \forall y) = \forall x \odot \forall y$,
M4°. $\forall(x \oplus x) = \forall x \oplus \forall x$,
M5°. $\forall(x \odot x) = \forall x \odot \forall x$.

A mapping $\forall : A \longrightarrow A$ satisfying the properties M0° - M5° will be called *universal quantifier* on A. A *monadic MV-algebra* is a pair $\langle A, \exists \rangle$ where A is an MV-algebra and $\exists$ is an existential quantifier on A. One can also define a monadic MV-algebra as a pair $\langle A, \forall \rangle$ where A is an MV-algebra and $\forall$ is an universal quantifier on A.

Lemma 2.4 [10] *In every monadic MV-algebra the following properties are satisfied:*
(i) $\exists 1 = 1$,
(ii) $\exists\exists x = \exists x$,
(iii) $\exists(\neg\exists x) = \neg\exists x$,
(iv) $\exists(\exists x \odot \exists y) = \exists x \odot \exists y$,
(v) $\exists(\exists x \oplus \exists y) = \exists x \oplus \exists y$,
(vi) $\exists(a \wedge \exists b) = \exists a \wedge \exists b$,
(vii) $\exists(a \vee b) = \exists a \vee \exists b$,
(viii) $x \leq y \Rightarrow \exists x \leq \exists y$ *and* $\forall x \leq \forall y$,
(ix) $\exists\forall x = \forall x$, $\forall\exists x = \exists x$.

Example 2.5 [3] If K is a non-empty set then $[0,1]^K$ becomes a monadic MV-algebra by defining $\exists : [0,1]^K \longrightarrow [0,1]^K$ in the following way:
$(\exists p)(k) = \bigvee\{p(l) \mid l \in K\}$ for any $p \in [0,1]^K$ and $k \in K$.
The axioms M0-M5 can be proved by using Lemma 2.1.

3 Monadic Pavelka algebras

Let us denote L the MV-algebra $[0,1] \cap \mathbb{Q}$.

Definition 3.1 *A* Pavelka algebra *is a structure* $\langle A, \{\overline{r} : r \in L\}\rangle$ *where* A *is an MV-algebra and* $\{\overline{r} : r \in L\} \subseteq A$ *such that:*
P0. $\overline{0} = 0$,
P1. $\overline{r \oplus s} = \overline{r} \oplus \overline{s}$ *for any* $r, s \in L$,
P2. $\overline{\neg r} = \neg\overline{r}$ *for any* $r \in L$,
P3. $\overline{r} \neq \overline{s}$ *for any distinct* $r, s \in L$.

Thus, the mapping $r \mapsto \overline{r}$ is an injective morphism of MV-algebras. The Lindenbaum - Tarski algebra of Rational Pavelka logic (RPL) is a Pavelka algebra. The notion of morphism of Pavelka algebras is introduced as usual.

Lemma 3.2 *Let* $\langle A, \{\overline{r} : r \in L\}\rangle$ *be a Pavelka algebra,* P *a proper filter of* A *and* $r, s \in L$. *Then the following hold:*
(i) $\overline{r} \in P$ *iff* $r = 1$,
(ii) $r \leq s$ *iff* $\overline{r}/_P \leq \overline{s}/_P$.

Proof. (i) If $r \neq 1$ then there is $n < \omega$ such that $r^n = 0$, so $\overline{r}^n = 0$. But $\overline{r} \in P$ implies $\overline{r}^m \in P$ for each $m < \omega$. We get $0 \in P$. Contradiction.
(ii) $r \leq s$ iff $r \longrightarrow s = 1$ iff $\overline{r \longrightarrow s} \in P$ iff $\overline{r} \longrightarrow \overline{s} \in P$ iff $\overline{r}/_P \leq \overline{s}/_P$. □

Definition 3.3 *A* monadic Pavelka algebra *is a structure* $\langle A, \exists, \{\overline{r} : r \in L\}\rangle$ *where* $\langle A, \exists\rangle$ *is a monadic MV-algebra and* $\langle A, \{\overline{r} : r \in L\}\rangle$ *is a Pavelka algebra such that* $\exists\overline{r} = \overline{r}$ *for any* $r \in L$.

The notion of morphism of monadic Pavelka algebras is introduced as usual.

Example 3.4 Let F be the set of formulas of Rational Pavelka predicate logic (RPL$\forall$) and $\sim$ the following equivalence relation on F: $\varphi \sim \psi$ iff $\vdash \varphi \leftrightarrow \psi$. If x is a variable then we denote $(\exists x)([\varphi]) = [\exists x\varphi]$ where φ is a formula and $[\varphi]$ its class in $F/_\sim$. Then $\langle F/_\sim, \exists x : F/_\sim \longrightarrow F_\sim\rangle$ is a monadic MV-algebra. If r, $s \in L$ are distinct then $[\overline{r}] \neq [\overline{s}]$. If $[\overline{r}] = [\overline{s}]$ then $\vdash \overline{r} \rightarrow \overline{s}$ and $\vdash \overline{s} \rightarrow \overline{r}$. We get $r \leq s$ and $s \leq r$ (see [4]), so $r = s$. It is easy to show that in this way $F/_\sim$ becomes a monadic Pavelka algebra.

Example 3.5 Let K be a non-empty set. For $r \in L$ denote $\overline{r} : K \longrightarrow [0,1]$ the constant function $k \mapsto r$. Thus $\left\langle [0,1]^K, \{\overline{r} : r \in L\}\right\rangle$ is a Pavelka algebra, so , by Example 2.5, $[0,1]^K$ is endowed with a structure of monadic Pavelka algebra.

If A is a monadic Pavelka algebra then a morphism of monadic Pavelka algebras $\Phi : A \longrightarrow [0,1]^K$ will be called a *representation* of A.

Lemma 3.6 *In a monadic Pavelka algebra* A *the following equalities hold for any* $r \in L$ *and* $a \in A$:
(i) $\exists(\overline{r} \oplus a) = \overline{r} \oplus \exists(a)$,
(ii) $\exists(\overline{r} \odot a) = \overline{r} \odot \exists(a)$,
(iii) $\forall(\overline{r} \oplus a) = \overline{r} \oplus \forall(a)$,
(iv) $\forall(\overline{r} \odot a) = \overline{r} \odot \forall(a)$,
(v) $\overline{r} \longrightarrow \exists a = \exists(\overline{r} \longrightarrow a)$,
(vi) $\exists a \longrightarrow \overline{r} = \forall(a \longrightarrow \overline{r})$,
(vii) $\overline{r} \longrightarrow \forall a = \forall(\overline{r} \longrightarrow a)$,
(viii) $\forall a \longrightarrow \overline{r} = \exists(a \longrightarrow \overline{r})$.

Proof. (i) $\exists(\overline{r} \oplus a) = \exists(\exists\overline{r} \oplus a) = \exists\overline{r} \oplus \exists a = \overline{r} \oplus \exists a$.
(ii), (iii), (iv) follows similarly.
(v) $\exists(\overline{r} \longrightarrow a) = \exists(\neg\overline{r} \oplus a) = \exists(\overline{\neg r} \oplus a) = \overline{\neg r} \oplus \exists a = \overline{r} \longrightarrow \exists a$.
(vi) $\forall(a \longrightarrow \overline{r}) = \forall(\neg a \oplus \overline{r}) = \overline{r} \oplus \forall\neg a = \overline{r} \oplus \neg\exists a = \exists a \longrightarrow \overline{r}$.
(vii), (viii) follows similarly. □

One remark that $B = \exists(A) = \forall(A)$ is a Pavelka subalgebra of A.

For the rest of the paper let $\langle A, \exists, \{\overline{r} : r \in L\}\rangle$ be an arbitrary monadic Pavelka algebra and $B = \exists(A)$.

Lemma 3.7 *If* $s \in L$, $a \in A$ *and* $\overline{s} \nleq a$ *then there exists* $X \subseteq B$ *such that:*
(i) $filt(X \cup \{a \longrightarrow \overline{s}\})$ *is proper,*
(ii) *for any* $b \in B$ *and* $r \in L$, $\overline{r} \longrightarrow b \in X$ *or* $b \longrightarrow \overline{r} \in X$.

Proof. We shall prove that the $filt(a \longrightarrow \overline{s})$ is proper. If not, then $(a \longrightarrow \overline{s})^n = 0$ for some $n < \omega$. But $(a \longrightarrow \overline{s})^n \vee (\overline{s} \longrightarrow a)^n = 1$ so $(\overline{s} \longrightarrow a)^n = 1$. This yields $\overline{s} \longrightarrow a = 1$, hence $\overline{s} \leq a$. Contradiction. Thus there exists $b \notin filt(a \longrightarrow \overline{s})$.

Consider an enumeration $\{(a_\xi, \overline{r}_\xi) \mid \xi < k\}$ of the set $B \times L$. We shall construct by induction a sequence $\{X_\xi\}_{\xi<k}$ such that $b \notin filt(X_\xi)$ for any $\xi < k$.

• $X_0 = \{a \longrightarrow \overline{s}\}$

• $\xi = \zeta + 1$. The induction hypothesis is $b \notin filt(X_\zeta)$. Assume $b \in filt(X_\zeta \cup \{a_\zeta \longrightarrow \overline{r}_\zeta\}) \cap filt(X_\zeta \cup \{\overline{r}_\zeta \longrightarrow a_\zeta\})$ so there is $n < \omega$ such that $(a_\zeta \longrightarrow \overline{r}_\zeta)^n \longrightarrow b \in filt(X_\zeta)$ and $(\overline{r}_\zeta \longrightarrow a_\zeta)^n \longrightarrow b \in filt(X_\zeta)$. But

$$\begin{aligned} b &= 1 \longrightarrow b \\ &= [(a_\zeta \longrightarrow \overline{r}_\zeta)^n \vee (\overline{r}_\zeta \longrightarrow a_\zeta)^n] \longrightarrow b \\ &= [(a_\zeta \longrightarrow \overline{r}_\zeta)^n \longrightarrow b] \wedge [(\overline{r}_\zeta \longrightarrow a_\zeta)^n \longrightarrow b] \end{aligned}$$

hence $b \in filt(X_\zeta)$. Contradiction. It follows that $b \notin filt(X_\zeta \cup \{a_\zeta \longrightarrow \overline{r}_\zeta\})$ or $b \notin filt(X_\zeta \cup \{\overline{r}_\zeta \longrightarrow a_\zeta\})$. Thus one can define

$$X_\xi = \begin{cases} X_\zeta \cup \{a_\zeta \longrightarrow \overline{r}_\zeta\} & \text{if } b \notin filt(X_\zeta \cup \{a_\zeta \longrightarrow \overline{r}_\zeta\}) \\ X_\zeta \cup \{\overline{r}_\zeta \longrightarrow a_\zeta\} & \text{otherwise.} \end{cases}$$

• If ξ is a limit ordinal then $X_\xi = \bigcup_{\zeta<\xi} X_\zeta$.

It follows that $b \notin filt(\bigcup_{\xi<k} X_\xi)$ and we define $X = \bigcup_{\xi<k} X_\xi - \{a \longrightarrow \overline{s}\}$. □

Lemma 3.8 *Let X be a the set constructed in Lemma 3.7. If $\exists a \in X$ then there exists a prime filter P such that $X \cup \{a\} \subseteq P$.*

Proof. By the dual of [2], Proposition 1.2.13 it suffices to prove that $filt(X \cup \{a\})$ is a proper filter. If not, then there exist $m < \omega$ and $x_1, \ldots, x_n \in X$ such that $x_1 \odot \cdots \odot x_n \odot a^m = 0$. Denote $x = x_1 \odot \cdots \odot x_n$ so $c \leq \neg a^m$, hence $\forall c \leq \forall\neg(a^m) = \neg\exists(a^m)$. But $c \in \exists(A)$ because $X \subseteq \exists(A)$ so $c \leq \neg\exists(a^m)$, hence $\neg\exists(a^m) \in filt(X)$. By hypothesis, $\exists(a^m) = (\exists a)^m \in filt(X)$, contradicting that $filt(X)$ is proper. □

4 Representation theorem

In this section we shall prove a representation theorem for monadic Pavelka algebras.

Theorem 4.1 *Let $\langle A, \exists, \{\overline{r} : r \in L\}\rangle$ be a monadic Pavelka algebra. If $a \in A$ and $s \in L$ such that $\overline{s} \not\leq a$ then there exist a non-empty set K, a representation $\Phi : A \longrightarrow [0,1]^K$ and $k \in K$ such that $\Phi(a)(k) \leq s$.*

Proof. Let X be the set constructed in Lemma 3.7 and K the set of prime filters of A including X. For any $x \in A$ and $P \in K$ denote

$$[x]_P = sup\{r \in L \mid \overline{r} \longrightarrow x \in P\}.$$

In order to define Φ we have to prove some properties.

(i) $[x]_P = inf\{r \in L \mid x \longrightarrow \overline{r} \in P\}$.

If $\overline{r} \longrightarrow x \in P$ and $x \longrightarrow \overline{s} \in P$ then, by Lemma 3.2, $\overline{r} \longrightarrow \overline{s} \in P$, so $r \leq s$. It follows that $[x]_P \leq inf\{r \in L \mid x \longrightarrow \overline{r} \in P\}$. If we assume $[x]_P < inf\{r \in L \mid x \longrightarrow \overline{r} \in P\}$ then there is $q \in L$ such that $[x]_P < q < inf\{r \in L \mid x \longrightarrow \overline{r} \in P\}$, so $\overline{q} \longrightarrow x \notin P$ and $x \longrightarrow \overline{q} \notin P$. This contradicts the fact that P is a prime filter.

(ii) $[x \oplus y]_P = [x]_P \oplus [y]_P$.

(iii) $[x \odot y]_P = [x]_P \odot [y]_P$.

In order to prove (ii) we have

$$[x \oplus y]_P = inf\{t \mid x \oplus y \longrightarrow \overline{t} \in P\} \text{ and}$$

$$[x]_P \oplus [y]_P = sup\{r \oplus q \mid \overline{r} \longrightarrow x \in P, \overline{q} \longrightarrow y \in P\}.$$

By Lemma 3.2, $\overline{r} \longrightarrow x \in P$, $\overline{q} \longrightarrow y \in P$ and $x \oplus y \longrightarrow \overline{t} \in P$ implies

$\overline{r}/_P \leq x/_P$, $\overline{q}/_P \leq y/_P$ and $x/_P \oplus y/_P \leq \overline{t}/_P$ so $\overline{r \oplus q}/_P \leq \overline{t}/_P$, hence $r \oplus q \leq t$. We proved that $[x]_P \oplus [y]_P \leq [x \oplus y]_P$. The converse inequality and (iii) follow similarly.
(iv) $[\overline{r}]_P = r$ for any $r \in L$.
By Lemma 3.2, $[\overline{r}] = sup\{q \in L \mid q \leq r\} = r$.

Let us define $\Phi : A \longrightarrow [0,1]^K$ by $\Phi(x)(P) = [x]_P$ for any $x \in A$ and $P \in K$. In accordance to (ii)-(iv), Φ is a morphism of Pavelka algebras. Now we shall prove that
(v) $\Phi(\exists x)(P) = (\exists \Phi(x))(P)$ for any $x \in A$ and $P \in K$.
If $r \in L$ and P, $Q \in K$ we have, in accordance to Lemma 3.2, $\overline{r} \longrightarrow \exists x \in P$ iff $\overline{r} \longrightarrow \exists x \in Q$, therefore $[\exists x]_P = [\exists x]_Q$. Then $[\exists x]_P = [\exists x]_Q \geq [x]_Q$ for every $Q \in K$, hence
$\Phi(\exists x)(P) = [\exists x]_P \geq sup\{[x]_Q \mid Q \in K\} = sup\{\Phi(x)(Q) \mid Q \in K\} = (\exists \Phi(x))(P)$.
The following implications:

$$\begin{aligned} r < [\exists x]_P &\Rightarrow \exists x \longrightarrow \overline{r} \notin P && \text{(cf. (i))} \\ &\Rightarrow \exists x \longrightarrow \overline{r} \notin X && \text{(cf. } X \subseteq P) \\ &\Rightarrow \overline{r} \longrightarrow \exists x \in X && \text{(cf. Lemma 3.7)} \\ &\Rightarrow \exists(\overline{r} \longrightarrow x) \in X && \text{(cf. Lemma 3.6)} \\ &\Rightarrow \overline{r} \longrightarrow x \in Q && \text{(cf. Lemma 3.8)} \\ &\Rightarrow r \leq [x]_Q \end{aligned}$$

for some $Q \in K$, establish the converse inequality in (v). Indeed, if we assume $[\exists x]_P > sup\{[x]_Q \mid Q \in K\}$ then there is $r \in L$ such that $[\exists x]_P > r > sup\{[x]_Q \mid Q \in K\}$ contradicting the above implications. Therefore, Φ is a representation of A.

Finally, by Lemma 3.7, there exists $P_0 \in K$ such that $X \cup \{a \longrightarrow \overline{s}\} \subseteq P_0$ so $\Phi(a)(P_0) \leq s$ and $\Phi(a)(P_0) = inf\{r \mid a \longrightarrow \overline{r} \in P_0\}$. □

For any $a \in A$ let us define
$[a] = sup\{r \mid \overline{r} \leq a\}$
$\| a \| = inf\{\Phi(a)(k) \mid \Phi : A \longrightarrow [0,1]^K$ representation and $k \in K\}$.

Corollary 4.2 $[a] = \| a \|$ *for any* $a \in A$.

Proof. The inequality $[a] \leq \| a \|$ is obvious. Assume there exists $s \in L$ such that $[a] < s < \| a \|$. Thus $\overline{s} \not\leq a$ so, by Theorem 4.1, there exist a representation $\Phi : A \longrightarrow [0,1]^K$ and $k \in K$ such that $\Phi(a)(k) \leq s$. Therefore $\| a \| \leq \Phi(a)(k) \leq s$. Contradiction. □

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