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Mixed Relations as Enriched Semiringal Categories[1]

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Abstract: A study of the classes of finite relations as enriched strict monoidal categories is presented in [CaS91]. The relations there are interpreted as connections in flowchart schemes, hence an "angelic" theory of relations is used. Finite relations may be used to model the connections between the components of dataflow networks [BeS98, BrS96], as well. The corresponding algebras are slightly different enriched strict monoidal categories modeling a "forward-demonic" theory of relations.
In order to obtain a full model for parallel programs one needs to mix control and reactive parts, hence a richer theory of finite relations is needed. In this paper we (1) define a model of such mixed finite relations, (2) introduce enriched (weak) semiringal categories as an abstract algebraic model for these relations, and (3) show that the initial model of the axiomatization (it always exists) is isomorphic to the defined one of mixed relations. Hence the axioms gives a sound and complete axiomatization for the these relations.

Key Words: parallel programs; mixed relations; network algebra; (enriched) semiringal category; abstract data type;
Category: F.3.2

1 Introduction

Powerful algebraic representations for logics are well-known; see, e.g., [Rud74]. In computer science, such a stage is still a desiderata. This paper is included in a series [Ste96a, Ste96b, Ste96c, GSB98, GBSS98] aiming to contribute to the algebraic theory of distributed computation. The key problem in understanding Multi-Agent Systems is to find a theory which integrates the *reactive* part and the *control* part of such systems. The claim of this series of papers is that the mixture of the additive and multiplicative network algebras (MixNA) will contribute to the understanding of distributed computation.

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The aim of the present paper is to give an algebraic presentation for finite relations suited to be used for modeling the interface changing connections between the statements/modules of parallel program schemes. The model for parallel programs we are using is MixNA, a mixture of additive and multiplicative network algebras presented in [Ste96a].

An extensive study of the classes of finite relations as enriched strict monoidal categories is presented in [CaS91]. The view taken in [CaS91] is to see relations as connections in (sequential) flowchart schemes, hence an "angelic" theory of relations is actually used there. Finite relations may be used to model the connections between the components of (parallel) dataflow networks [BeS98, BrS96], as well. The corresponding algebras are slightly different enriched strict monoidal categories modeling a "forward-demonic" theory of relations. Similar algebraic structures where recently used in [Gib95] to handle acyclic graphs, or in [BGM98] to get a unifying framework for representing "concurrent and distrbuted systems."

In order to obtain a full model for parallel programs one needs to mix control and reactive parts, hence a richer theory of finite relations is needed. We introduced enriched (weak) semiringal categories as an algebraic model for relations in this mixed setting.

The main result of the paper consists of presenting the mixed relations as an initial model in the category of enriched semiringal categories. The relations corresponding to a particular choice (x_1, x_2, x_3, x_4) with $x_i \in \{a, b, c, d\}$, are called $x_1x_2x_3x_4$-relations. As a consequence of this result we get axiomatizations for most of the resulting 256 types of interface-changing relations. (Some cases with $x_4 \in \{c, d\}$ are open).

2 Preliminaries

In this section we present the definition of symmetric semiringal categories. This structure was introduced, e.g., in [Ste96a, Ste98]. The present definition is given long the line suggested in [Luc99].

A *symmetric semiringal category* $(M, \oplus, \otimes, \cdot, \mathsf{I}, \curlyvee)$ is a mixture between two symmetric strict monoidal categories $(M, \oplus, \cdot, \mathsf{I}, \curlyvee)$ and $(M, \otimes, \cdot, \mathsf{I})$ where $\otimes$ distributes over $\oplus$ and the zero-rules for I_0 hold. In order to express these additional properties we use two new types of constants: $\delta_{a;b,c} : a \otimes (b \oplus c) \cong (a \otimes b) \oplus (a \otimes c)$, denoting the isomorphisms corresponding to the distribution of products over sums of objects, and $\delta_{a;} : a \otimes 0 \cong 0$, denoting the isomorphism between the product $a \otimes 0$ and 0. These new constants must satisfy the axioms D1-D10 listed in the appendix. We will see later how the constants δ are generalized to more complex terms.

If C is a subset of morphisms in M, then we say $(M, \oplus, \otimes, \cdot, \mathsf{I}, \delta, \rho, \curlyvee, \mathsf{X})$ is a C-*weak* symmetric semiringal category over S if the strong distributivity axioms D9-D10 are required for morphisms f in C, only (and arbitrary g, h in M).

Various enriched symmetric semiringal categories are obtained adding additive or multiplicative branching constants: ${}_a\!\bullet\!\!<_k, k \geq 0$ (additive ramification); ${}_k\!>\!\!\bullet_a, k \geq 0$ (additive identification); $\wedge^a_k, k \geq 0$ (multiplicative ramification); $\vee^k_a, k \geq 0$ (multiplicative identification). These constants must satisfy the axioms SV1-20, AddS1-4, and MultS1-4 included in the appendix. As is [CaS91], the restrictions a $(k = 1)$, b $(k \leq 1)$, c $(k \geq 1)$, and d (arbitrary k) may be freely used for each type of branching constant ${}_a\!\bullet\!\!<_k$, ${}_k\!>\!\!\bullet_a$, $\curlywedge^a_k$, or $\curlyvee^k_a$. A $x_1x_2x_3x_4$-*enriched semiringal category* is an enriched semiringal category where the use of branching constants is restricted in concordance with $x_1x_2x_3x_4$. Obviously, an $aaaa$-enriched semiringal category is just a symmetric semiringal category.

3 Mixed relations

Let S be a set of atomic sorts. The set of *sort terms* is obtained with the rules $a = a \oplus a \mid a \otimes a \mid 0 \mid 1 \mid s(\in S)$. In the rest of this paper we use the following notations:

- ka for $a \oplus \cdots \oplus a$ (k times); if $k = 0$ then $ka = 0$.
- a^k for $a \otimes \cdots \otimes a$ (k times); if $k = 0$ then $a^k = 1$.

The set of *sorts* $S^{\oplus,\otimes}$ is the set of sort terms modulo the congruence generated by the following axioms:

$$\begin{aligned} &a \oplus (b \oplus c) = (a \oplus b) \oplus c,\\ &a \otimes (b \otimes c) = (a \otimes b) \otimes c,\\ &a \oplus 0 = a,\\ &a \otimes 1 = a. \end{aligned} \tag{1}$$

Each equivalence class is represented by a unique *flattened term* obtained by removing the parenthesis due to the associativity axioms and deleting the constants 0 and 1 accordingly with the last two axioms. More precisely, a flattened term is an irreducible element relative to the following confluent and terminating rewriting system:

$$\begin{aligned} &a \oplus (b \oplus c) \to a \oplus b \oplus c,\\ &(a \oplus b) \oplus c \to a \oplus b \oplus c,\\ &a \otimes (b \otimes c) \to a \otimes b \otimes c,\\ &(a \otimes b) \otimes c \to a \otimes b \otimes c,\\ &a \oplus 0 \to a,\\ &a \otimes 1 \to a. \end{aligned} \tag{2}$$

A sort is often identified with the flattened element representing the class. For a sort a the associated (unique) flattened element is denote by fl_a.

Definition 1. Let a be a sort term. The set $\mathsf{Pos}(a) \subset \omega^*$ of *positions* in a, and the set $\mathsf{Fr}(a) \subset \omega^*$ of *frontier positions* in a, and the function $\overline{a} : \mathsf{Pos}(a) \to S \cup \{\oplus, \otimes, 1, 0\}$ are inductively defined as follows:

- if $a \in \{0, 1\}$ then $\mathsf{Pos}(a) = \mathsf{Fr}(a) = \emptyset$;
- if $a \in S$ then $\mathsf{Pos}(a) = \mathsf{Fr}(a) = \{\varepsilon\}$ (ε denotes the empty string) and $\overline{a}(\varepsilon) = a$;
- if $a = a_1 \text{ op } a_2$ with op $\in \{\oplus, \otimes\}$, then:
  - $\mathsf{Pos}(a) = \{\varepsilon\} \cup \{i.p \mid p \in \mathsf{Pos}(a_i), i = 1, 2\}$
    ($\_.\_$ denotes the concatenation operation on ω^*),
  - $\mathsf{Fr}(a) = \{i.p \mid p \in \mathsf{Fr}(a_i), i = 1, 2\}$, and
  - $\overline{a}(\varepsilon) = \text{op}$, $\overline{a}(i.p) = a_i(p)$ for all $p \in \mathsf{Pos}(a_i)$, $i = 1, 2$.

The definition given above is extended to the flattened terms in the usual way:

– if $a = a_1 \text{ op} \cdots \text{op } a_n$ with op $\in \{\oplus, \otimes\}$, then:
  - $\mathsf{Pos}(a) = \{\varepsilon\} \cup \{i.p \mid p \in \mathsf{Pos}(a_i), i \in [n]\}$,
  - $\mathsf{Fr}(a) = \{i.p \mid p \in \mathsf{Fr}(a_i), i \in [n]\}$, and
  - $\overline{a}(\varepsilon) = \text{op}$, $\overline{a}(i.p) = a_i(p)$ for all $p \in \mathsf{Pos}(a_i)$, $i \in [n]$.

If a is a sort term and fl_a is the flattened term equivalent to a, then there is bijection

$$\Downarrow_a : \mathsf{Pos}(a) \to \mathsf{Pos}(\mathsf{fl}_a)$$

which is uniquely determined rewriting rules (2). For example, if $a = a_1 \otimes (a_2 \otimes a_3)$ and $\mathsf{fl}_a = a_1 \otimes a_2 \otimes a_3$ then $\Downarrow_a : \{1, 2.1, 2.2\} \to \{1, 2, 3\}$ is given by $\Downarrow_a(1) = 1, \Downarrow_a(2.1) = 2, \Downarrow_a(2.2) = 3$. When the subscript a may be derived from the context we simply write $\Downarrow p$ instead of $\Downarrow_a(p)$, where $p \in \mathsf{Pos}(a)$.

Definition 2. Consider a a sort term and p and q two frontier positions in $\mathsf{Fr}(a)$. We say that *p and q coexist* (in a) iff (1) $p = q$ or (2) $\overline{a}(\text{lcp}(p, q)) = \otimes$, where $\text{lcp}(p, q)$ denotes the longest common prefix of p and q. We denote by $\_ \overset{co}{\sim} \_$ the coexistence relation.
A set consisting of frontier positions which simultaneously coexist in a sort a is called a *multiplicative world* of a.

> A sort describe the interface of a parallel program. The intuition behind the above definition is that the frontier positions of a multiplicative world describe the components of a state of the system. If two positions do not coexist, then they belong to disjoint (or incompatible) states.

Definition 3. Consider a and b two sorts in $S^{\oplus,\otimes}$. A *pure multiplicative relation of type $a \to b$* is a sequence of maps $(p_1 \mapsto q_1, \ldots, p_k \mapsto q_k)$ where $p_i \in \mathsf{Fr}(a)$, $q_i \in \mathsf{Fr}(b)$, $i = 1, \ldots, k$, such that

– $\overline{a}(p_i) = \overline{a}(q_i)$ for $i = 1, \ldots, k$, and
– $p_i \overset{co}{\sim} p_j$ and $q_i \overset{co}{\sim} q_j$ for all $i, j = 1, \ldots k$ (in other words, $\{p_1, \ldots, p_k\}$ and $\{q_1, \ldots, q_k\}$ are multiplicative worlds of a and b, respectively).

A *mixed relation of type $a \to b$* is a set of pure multiplicative relations of type $a \to b$. $\mathsf{MixRel}_S(a, b)$ denotes the set of all mixed relations of type $a \to b$, and $x_1x_2x_3x_4$-$\mathsf{MixRel}_S(a, b)$ denotes the subset of mixed relations of type $a \to b$ which satisfy $x_1x_2x_3x_4$-restrictions on branching constants.

We use the notation $f = \{f_1 \mid f_2 \mid \cdots \mid f_n\}$ for a mixed relation f in order to emphasize the additive character of f. The above notation can be thought as the Backus-Naur rule

$$f ::= f_1 \mid f_2 \mid \cdots \mid f_n.$$

The order of maps of a pure multiplicative relation is not important. Also, any map can occur only once into a pure multiplicative relation. We use the notation $p \mapsto q \in f$ whenever we want to say that $p \mapsto q$ occurs in a pure multiplicative component of f.

Example 1. Consider $a = (a_1 \otimes a_2) \oplus (a_1 \otimes a_3)$, $b = a_1 \otimes (a_2 \oplus a_1)$ where $a_i \in S$, $i = 1, 2, 3$. Then:

– the frontier sets are $\mathsf{Fr}(a) = \{1.1, 1.2, 2.1, 2.2\}$ and $\mathsf{Fr}(b) = \{1, 2.1, 2.2\}$;

– the coexistence relation in a is $\{1.1 \overset{co}{\approx} 1.2,\ 2.1 \overset{co}{\approx} 2.2\}$ and the coexistence relation in b is $\{1 \overset{co}{\approx} 2.1,\ 1 \overset{co}{\approx} 2.2\}$; and
– $f = \{(1.1 \mapsto 1,\ 1.2 \mapsto 2.1) \mid (2.1 \mapsto 2.2)\}$ is a mixed relation. It may be graphically represented as in Fig. 1. The first pure multiplicative component is drawed using solid lines and the second component using dashed lines.

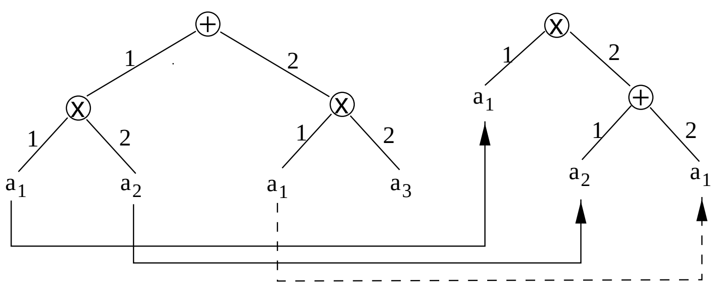


Figure 1: A mixed relation

We often write $\{\cdots\}_{a\to b}$ to emphasize that $\{\cdots\} \in \mathsf{MixRel}_S(a,b)$.

Definition 4. Consider $f : a \to b$ and $g : b \to c$ two pure multiplicative relations. Then the *(forward demonic) sequential composition* of f and g is the relation $f \cdot g = (p \mapsto r \mid (\exists q.\ p \mapsto q \in f$ and $q \mapsto r \in g)$ and $(\forall q'.\ q' \mapsto r \in g\ \exists p'.\ p' \mapsto q' \in f))$.

Remark. Accordingly to the demonic/angelic dichotomy we may also consider the angelic sequential composition of the pure multiplicative relations as being the usual composition of relations. This kind of composition is not of interest for this paper. Actually, the angelic and demonic sequential compositions coincide in the case no equality test $\forall_a^k$ with $k \geq 2$ is present.

Before defining the operations on mixed relations, a remark concerning the flattened sorts is necessary.

Remark. Recall that a sort is identified to the flattened element representing the equivalence class defining the sort. So, in the definition(s) to follow, a, b, and c are flattened elements. But not always from the fact that a and b are flattened elements follows that $a \oplus b$ and $a \otimes b$ are flattened elements and therefore the flattening functions $\Downarrow$ are to be used. The same thing is true for the definition of $[\![t]\!]$ we shall give later.

Definition 5. The operations $\_\oplus\_$, $\_\otimes\_$, and $\_\cdot\_$ over mixed relations are defined as follows:

1 Suppose $f : a \to b,\ g : a' \to b'$. Then $f \oplus g : a \oplus a' \to b \oplus b'$ is given by
$f \oplus g = \{(\Downarrow_{a\oplus a'}(1.p) \mapsto \Downarrow_{b\oplus b'}(1.q) \mid p \mapsto q \in f_i) \mid i \in [m]\}$
$\cup \{(\Downarrow_{a\oplus a'}(2.p) \mapsto \Downarrow_{b\oplus b'}(2.q) \mid p \mapsto q \in g_j) \mid j \in [n]\}$
where $f = \{f_1 \mid \ldots \mid f_m\}$ and $g = \{g_1 \mid \ldots \mid g_n\}$.

2. Suppose $f : a \to b,\ g : a' \to b'$. Then $f \otimes g : a \otimes a' \to b \otimes b'$ is given by
$f \otimes g = \{(\Downarrow_{a\otimes a'}(1.p) \mapsto \Downarrow_{b\otimes b'}(1.q), \Downarrow_{a\otimes a'}(2.p') \mapsto \Downarrow_{b\otimes b'}(2.q') \mid$
$p \mapsto q \in f_i, p' \mapsto q' \in g_j) \mid i \in [m],\ j \in [n]\}$
where $f = \{f_1 \mid \ldots \mid f_m\}$ and $g = \{g_1 \mid \ldots \mid g_n\}$.

3. Suppose $f : a \to b,\ g : b \to c$. Then $f \cdot g : a \to c$ is given by
$f \cdot g = \{f_i \cdot g_j \mid i \in [m],\ j \in [n]\}$
where $f = \{f_1 \mid \ldots \mid f_m\}$ and $g = \{g_1, \ldots, g_n\}$.

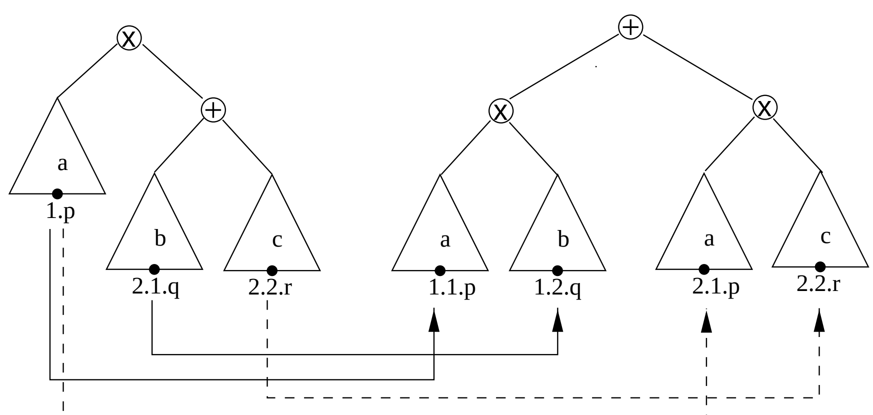
A mixed relation $f = \{(p_{11} \mapsto q_{11}, \ldots, p_{1n(1)} \mapsto q_{1n(1)}) \mid \ldots \mid (p_{m1} \mapsto q_{m1}, \ldots, p_{mn(m)} \mapsto q_{mn(m)})\}$ is often denoted by the simpler notation $\{\!|p_{11} \mapsto q_{11}, \ldots, p_{mn(m)} \mapsto q_{mn(m)}|\!\}$ provided the pure multiplicative components of f are uniquely determined by the maximal subsets of coexisting positions. For example, if $a = b = (a_1 \oplus a_2) \otimes (a_3 \oplus a_4)$ and $a_i \in S$ for all $i = 1, \ldots, 4$, then $\{\!|p \mapsto p \mid p \in \mathsf{Fr}(a)|\!\}$ uniquely denotes the mixed relation $\{(1.1 \mapsto 1.1, 2.1 \mapsto 2.1) \mid (1.1 \mapsto 1.1, 2.2 \mapsto 2.2) \mid (1.2 \mapsto 1.2, 2.1 \mapsto 2.1) \mid (1.2 \mapsto 1.2, 2.2 \mapsto 2.2)\}$. This convention is intensively used in the definition of $[\![t]\!]$ given below. Moreover, if $f = \{\!|p_i \mapsto q_i \mid i \in [n]|\!\} = \{f_1 \mid \ldots \mid f_k\}$, $g = \{\!|p'_i \mapsto q'_i \mid i \in [n']|\!\} = \{g_1 \mid \cdots \mid g_\ell\}$, $h = \{\!|p_1 \mapsto q_1, \ldots, p_n \mapsto q_n, p'_1 \mapsto q'_1, \ldots, p'_{n'} \mapsto q'_{n'}\} = \{f_1 \mid \cdots \mid f_k \mid g_1 \mid \cdots \mid g_\ell\}$ then we simply write $h = f \cup g$.

The set $\Sigma = \{\oplus, \otimes, \cdot, \mathsf{I}, \delta, \chi, \bullet\!\!<, >\!\!\bullet, \mathsf{X}, \curlywedge, \curlyvee\}$ is in fact a $S^{\oplus,\otimes} \times S^{\oplus,\otimes}$-sorted signature and a ground Σ-term will be called a *mixed term*. Each mixed term t defines a mixed relation $[\![t]\!]$ as follows:

0. $[\![\mathsf{I}_0]\!] = \{\ \}_{0\to 0}$ and $[\![\mathsf{I}_1]\!] = \{(\,)\}_{1\to 1}$.

1. $t = \mathsf{I}_a : a \to a$. Then $[\![t]\!] = \{\!|p \mapsto p \mid p \in \mathsf{Fr}(a)|\!\}$.
Examples: $[\![\mathsf{I}_{a_1\oplus a_2}]\!] = \{(1 \mapsto 1) \mid (2 \mapsto 2)\}$, $[\![\mathsf{I}_{a_1\otimes a_2}]\!] = \{(1 \mapsto 1, 2 \mapsto 2)\}$ where $a_1, a_2 \in S$.

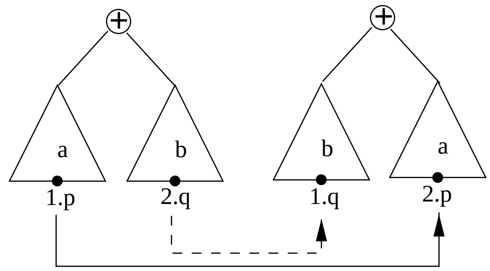
2. $t = \delta_{a;b,c} : a \otimes (b \oplus c) \to (a \otimes b) \oplus (a \otimes c)$. Then $[\![t]\!] =$
$\{\!|\Downarrow_{a\otimes(b\oplus c)}(1.p) \mapsto \Downarrow_{(a\otimes b)\oplus(a\otimes c)}(1.1.p),$
$\Downarrow_{a\otimes(b\oplus c)}(1.p) \mapsto \Downarrow_{(a\otimes b)\oplus(a\otimes c)}(2.1.p),$
$\Downarrow_{a\otimes(b\oplus c)}(2.1.q) \mapsto \Downarrow_{(a\otimes b)\oplus(a\otimes c)}(1.2.q),$
$\Downarrow_{a\otimes(b\oplus c)}(2.2.r) \mapsto \Downarrow_{(a\otimes b)\oplus(a\otimes c)}(2.2.r) \mid p \in \mathsf{Fr}(a), q \in \mathsf{Fr}(b), r \in \mathsf{Fr}(c)|\!\}$.



Example: $[\![\delta_{a_1;b_1\otimes b_2,c_1\oplus c_2}]\!] = \{(1 \mapsto 1.1,\ 2.1.1 \mapsto 1.2,\ 2.1.2 \mapsto 1.3) \mid (1 \mapsto 2.1,\ 2.2 \mapsto 2.2.1) \mid (1 \mapsto 2.1,\ 2.3 \mapsto 2.2.2)\}$ where $a_1, b_1, b_2, c_1, c_2 \in S$.

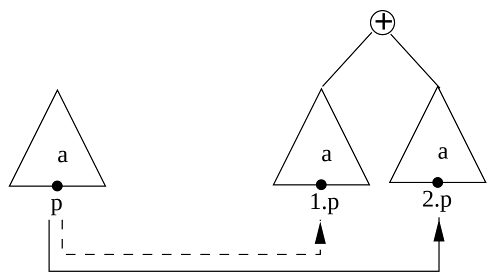
3. $[\![\delta_{a;}]\!] = \{\ \}_{a\otimes 0\to 0}$.

4. $t = {}^{b}_{a}\!\rangle\!\langle : a \oplus b \to b \oplus a$. Then $[\![t]\!] = \{\!|\Downarrow_{a\oplus b}(1.p) \mapsto \Downarrow_{b\oplus a}(2.p), \Downarrow_{a\oplus b}(2.q) \mapsto \Downarrow_{b\oplus a}(1.q) \mid p \in \mathsf{Fr}(a), q \in \mathsf{Fr}(b)|\!\}$.



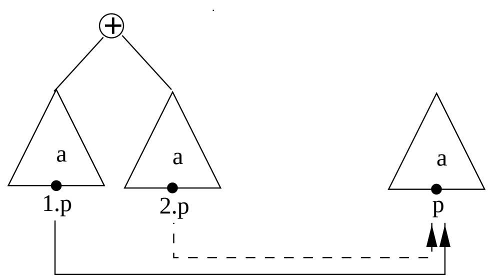
Example: $[\![{}^{a_3\oplus a_4}_{a_1\otimes a_2}\!\rangle\!\langle]\!] = \{(1.1 \mapsto 3.1, 1.2 \to 3.2) \mid (2 \mapsto 1) \mid (3 \mapsto 2)\}$ where $a_1, \ldots, a_4 \in S$.

5. $t = {}_a\!\bullet\!\!<_k : a \to ka$. Then $[\![t]\!] = \cup_{1\le i\le k}\{\!|p \mapsto \Downarrow_{ka}(i.p) \mid p \in \mathsf{Fr}(a)|\!\}$.



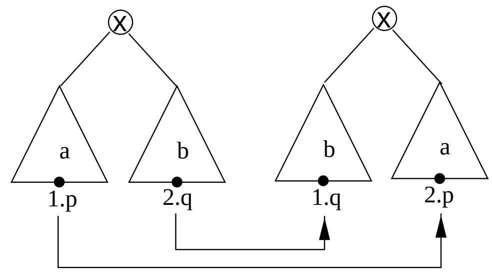
Examples: $[\![{}_{a_1\otimes a_2}\!\bullet\!\!<_2]\!] = \{(1 \mapsto 1.1, 2 \mapsto 1.2) \mid (1 \mapsto 2.1, 2 \mapsto 2.2)\}$ and $[\![{}_{a_1\oplus a_2}\!\bullet\!\!<_2]\!] = \{(1 \mapsto 1) \mid (1 \mapsto 3) \mid (2 \mapsto 2) \mid (2 \mapsto 4)\}$ where $a_1, a_2 \in S$.

6. $t = {}_k\!>\!\!\bullet_a : ka \to a$. Then $[\![t]\!] = \cup_{1\le i\le k}\{\!|\Downarrow_{ka}(i.p) \mapsto p \mid p \in \mathsf{Fr}(a)|\!\}$.



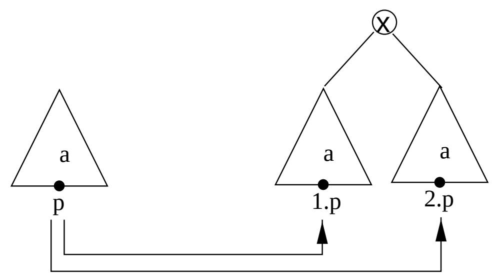
Examples: $[\![{}_2\!>\!\!\bullet_{a_1\otimes a_2}]\!] = \{(1.1 \mapsto 1, 1.2 \mapsto 2) \mid (2.1 \mapsto 1, 2.2 \mapsto 2)\}$ and $[\![{}_2\!>\!\!\bullet_{a_1\oplus a_2}]\!] = \{(1 \mapsto 1) \mid (3 \mapsto 1) \mid (2 \mapsto 2) \mid (4 \mapsto 2)\}$ where $a_1, a_2 \in S$.

7. $t = {}^a\mathsf{X}^b : a \otimes b \to b \otimes a$. Then $[\![t]\!] = \{\!|\Downarrow_{a\otimes b}(1.p) \mapsto \Downarrow_{b\otimes a}(2.p), \Downarrow_{a\otimes b}(2.q) \mapsto \Downarrow_{b\otimes a}(1.q) \mid p \in \mathsf{Fr}(a), q \in \mathsf{Fr}(b)|\!\}$.



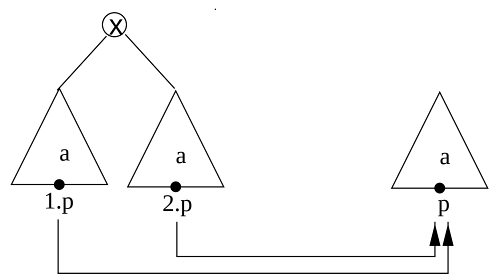
Example: $[\![{}^{a_1 \oplus a_2}\mathsf{X}^{a_3 \otimes a_4}]\!] = \{(1.1 \mapsto 3.1,\ 2 \mapsto 1,\ 3 \mapsto 2) \mid (1.2 \mapsto 3.2,\ 2 \mapsto 1,\ 3 \mapsto 2)\}$ where $a_1, \dots, a_4 \in S$.

8. $t = \curlywedge_k^a : a \to a^k$. Then $[\![t]\!] = \{\!|p \mapsto \Downarrow_{a^k}(i.p) \mid p \in \mathsf{Fr}(a), i = 1, \dots, k|\!\}$.



Examples: $[\![\curlywedge_2^{a_1 \oplus a_2}]\!] = \{(1 \mapsto 1.1,\ 1 \mapsto 2.1) \mid (2 \mapsto 1.2,\ 2 \mapsto 2.2)\}$ and $[\![\curlywedge_2^{a_1 \otimes a_2}]\!] = \{(1 \mapsto 1,\ 1 \mapsto 3,\ 2 \mapsto 2,\ 2 \mapsto 4)\}$ where $a_1, a_2 \in S$.

9. $t = \curlyvee_a^k : a^k \to a$. Then $[\![t]\!] = \{\!|\Downarrow_{a^k}(i.p) \mapsto p \mid p \in \mathsf{Fr}(a), i = 1, \dots, k|\!\}$.



Examples: $[\![\curlyvee^2_{a_1 \oplus a_2}]\!] = \{(1.1 \mapsto 1,\ 2.1 \mapsto 1) \mid (1.2 \mapsto 2,\ 2.2 \mapsto 2)\}$ and $[\![\curlyvee^2_{a_1 \otimes a_2}]\!] = \{(1 \mapsto 1,\ 3 \mapsto 1,\ 2 \mapsto 2,\ 4 \mapsto 2)\}$ where $a_1, a_2 \in S$.

10. $t = t_1 \oplus t_2$. Then $[\![t]\!] = [\![t_1]\!] \oplus [\![t_2]\!]$.
Example: $[\![\mathsf{I}_a \oplus {}^b\mathsf{X}^c]\!] = \{(1 \mapsto 1) \mid (2.1 \mapsto 2.2,\ 2.2 \mapsto 2.1)\}$ because $[\![\mathsf{I}_a]\!] = \{(\varepsilon \mapsto \varepsilon)\}$ and $[\![{}^b\mathsf{X}^c]\!] = \{(1 \mapsto 2,\ 2 \mapsto 1)\}$, where $a, b, c \in S$.

11. $t = t_1 \otimes t_2$. Then $[\![t]\!] = [\![t_1]\!] \otimes [\![t_2]\!]$.
Example: $[\![{}^b_a\mathsf{X} \otimes {}^b_a\mathsf{X}]\!] = \{(1.1 \mapsto 1.2,\ 2.1 \mapsto 2.2) \mid (1.1 \mapsto 1.2,\ 2.2 \mapsto 2.1) \mid (1.2 \mapsto 1.1,\ 2.1 \mapsto 2.2) \mid (1.2 \mapsto 1.1,\ 2.2 \mapsto 2.1)$ where $a, b \in S$.

12. $t = t_1 \cdot t_2$. Then $[\![t]\!] = [\![t_1]\!] \cdot [\![t_2]\!]$.
Example: $[\![(\mathsf{I}_a \otimes {}^c_b\mathsf{X}) \cdot \delta_{a;c,b}]\!] = \{(1 \mapsto 2.1,\ 2.1 \mapsto 2.2) \mid (1 \mapsto 1.1,\ 2.2 \mapsto 1.2)\}$

because $[\![\mathsf{I}_a \otimes {}^c_b\mathsf{X}]\!] = \{(1 \mapsto 1,\ 2.1 \mapsto 2.2) \mid (1 \mapsto 1,\ 2.2 \mapsto 2.1)\}$ and $[\![\delta_{a;c,b}]\!] = \{(1 \mapsto 1.1,\ 2.1 \mapsto 1.2) \mid (1 \mapsto 2.1,\ 2.2 \mapsto 2.2)\}$, where $a, b, c \in S$.

The distributivity constants $\delta_{\_;\_,\_}$ and $\delta_{\_;}$ can be extended to arbitrary number of arguments as follows:

$\delta_{a;b} = \delta_{a;b,0} = \mathsf{I}_{a\otimes b}$,
$\delta_{a;b_1,\ldots,b_{n+1}} = \delta_{a;b_1\oplus\cdots\oplus b_n,b_{n+1}} \cdot (\delta_{a;b_1,\ldots,b_n} \oplus \mathsf{I}_{a\otimes b_{n+1}})$,
$\delta_{a,b;c} = {}^{a\oplus b}\mathsf{X}^c \cdot \delta_{c;a,b} \cdot ({}^c\mathsf{X}^a \oplus {}^c\mathsf{X}^b)$,
$\delta_{a_1,\ldots,a_{m+1};b} = \delta_{a_1\oplus\cdots\oplus a_m,a_{m+1};b} \cdot (\delta_{a_1,\ldots,a_m;b} \oplus \mathsf{I}_{a_{m+1}\otimes b})$,
$\delta_{a_1,\ldots,a_m;b_1,\ldots,b_n} = \delta_{a_1,\ldots,a_m;b_1\oplus\cdots\oplus b_n} \cdot (\delta_{a_1;b_1,\ldots,b_n} \oplus \cdots \oplus \delta_{a_m;b_1,\ldots,b_n})$,
$\delta_{\mathbf{a}_1;\ldots;\mathbf{a}_m;\mathbf{b}} = ((\mathsf{I}_{\oplus\mathbf{a}_1} \otimes \cdots \otimes \mathsf{I}_{\oplus\mathbf{a}_{m-1}}) \oplus \delta_{\mathbf{a}_m;\mathbf{b}}) \cdot \delta_{\mathbf{a}_1;\cdots;\mathbf{a}_{m-1};a_{m1}\otimes b_1,\ldots,a_{mn(m)}\otimes b_n}$
where $\mathbf{a}_i = a_{i1},\ldots,a_{in(i)}$ for $i = 1,\ldots,m$, $\mathbf{b} = b_1,\ldots,b_n$, and $\oplus\mathbf{a}_i = a_{i1} \oplus \cdots \oplus a_{in(i)}$ for $i = 1,\ldots,m$.

Definition 6. The *additive (mixed) normal form* nf_a and the mixed term dis_a, for each $a \in S^{\oplus,\otimes}$, are inductively defined as follows:

– if $a \in S \cup \{0, 1\}$ then $\mathsf{nf}_a = a$ and $\mathsf{dis}_a = \mathsf{I}_a$;
– if $a = a_1 \oplus \cdots \oplus a_m$ then $\mathsf{nf}_a = \mathsf{nf}_{a_1} \oplus \cdots \oplus \mathsf{nf}_{a_m}$ and $\mathsf{dis}_a = \mathsf{dis}_{a_1} \oplus \cdots \oplus \mathsf{dis}_{a_m}$;
– if $a = a_1 \otimes \cdots \otimes a_m$ then
  - $\mathsf{nf}_a = \bigoplus_{(j(1),\ldots,j(m))}(b_{1j(1)} \otimes \cdots \otimes b_{mj(m)})$ and
  - $\mathsf{dis}_a = (\mathsf{dis}_{a_1} \otimes \cdots \otimes \mathsf{dis}_{a_m}) \cdot \delta_{\mathbf{b}_1;\cdots;\mathbf{b}_m;}$,
  where $\mathsf{nf}_{a_i} = b_{i1} \oplus \cdots \oplus b_{in(i)}$, $(j(1),\ldots,j(m)) \in [n(1)] \times \cdots [n(m)]$ are listed in lexicographic order, and $\mathbf{b}_i = b_{i1},\ldots,b_{in(i)}$ for $i = 1,\ldots,m$.

Remark. The additive normal form of a sort a is a sum of products. For example, if $a = (a_1 \oplus a_2) \otimes (a_3 \oplus a_4)$ then $\mathsf{nf}_a = (a_1 \otimes a_3) \oplus (a_1 \otimes a_4) \oplus (a_2 \otimes a_3) \oplus (a_2 \otimes a_4)$ and $\mathsf{dis}_a = \delta_{a_1,a_2;a_3,a_4}$ ($a_i \in S$ for $i = 1,\ldots,4$). It is easy to check that if $p \mapsto q,\ p' \mapsto q' \in [\![\mathsf{dis}_a]\!]$ then $p \overset{co}{\sim} p'$ in a iff $q \overset{co}{\sim} q'$ in nf_a. This is equivalent to say that the relation $[\![\mathsf{dis}_a]\!]$ preserves the maximal multiplicative worlds.

Example 2. Normal form interfaces and mixrelations between these interfaces may be described in a programming-like notation as follows. Take the following interfaces:

```
I:Interface Things of
      1:Class Objects of
            Elms 1:ball, 2:cub, 3:apple;
      2:Class Colours of
            Elms 1:red, 2:blue, 3:black;
      3:Class Weights of
            Elms 1:heavy, 2:light, 3:medium;
End Interface
```

```
J:Interface Things_with_attributes of
      1:Class Coloured_Objects of
            Elms 1:red_ball, 2:blue_ball, 3:red_apple, 4:black_cub;
      2:Class Weighted_Objects of
            Elms 1:heavy_cub, 2:medium_apple, 3:heavy_ball;
End Interface
```

We describe a $a\delta d\alpha$ mixrelation (a forward function on additive terms and backward functions on each connected multiplicative subterms). E.g., $\phi = (f; f^1, f^2, f^3) : \mathtt{I} \to \mathtt{J}$ defined by

$$\phi = \begin{array}{c|l} i & 1\ 2\ 3\ 4 \\ \hline f(i) & 1\ 1\ 2 \\ f^1(i) & 1\ 1\ 3\ 2 \\ f^2(i) & 1\ 2\ 1\ 3 \\ f^3(i) & 1\ 3\ 1 \end{array}$$

In this particular case, a mixfunction is given by: (1) a forward function f between the terms in the sum (the one which maps both `Objects` and `Colours` in `Coloured_Objects` and `Weights` in `Weighted_Objects`), and (2) for each connected monommials ("classes") a backword function, i.e., f^1 specifies the object in `Objects` associated to a coloured object in `Coloured_Objects`), f^2 does the same for the colours associated to coloured objects, and finally f^3 gives the weights associated to weighted objects.

4 Axiomatization

In this section we prove that the category whose objects are sorts and whose arrows are mixed relations is initial in the category of enriched symmetric semiringal categories.

Proposition 7. *Let E denote the set of axioms B1-2, AddB3-10, MultB3-10, D1-10. If $t : a \to \mathsf{nf}_a$ is a mixed $\{\oplus, \otimes, \cdot, \mathsf{I}, \delta, \curlywedge, \mathsf{X}\}$-term such that $[\![t]\!] = [\![\mathsf{dis}_a]\!]$ then $E \vdash t = \mathsf{dis}_a$.*

Proof. This is a particular case of Theorem 2 in [Luc99]. □

Lemma 8. *Let a and b two sorts in additive normal forms. For each f in $\mathsf{MixRel}_S(a, b)$ there is a mixed term t such that $[\![t]\!] = f$.*

Proof. Suppose that $a = a_1 \oplus \cdots \oplus a_m$, $b = b_1 \oplus \cdots \oplus b_n$, $f = \{(p_{11} \mapsto q_{11}, \ldots, p_{1n(1)} \mapsto q_{1n(1)}) \mid \ldots \mid (p_{m1} \mapsto q_{m1}, \ldots, p_{mn(m)} \mapsto q_{mn(m)})\}$. Because $(p_{i1} \mapsto q_{i1}, \ldots, p_{in(i)} \mapsto q_{in(i)})$ is a pure multiplicative relation it follows that all p_{ij} for $j = 1, \ldots, n(i)$ are in the same multiplicative component $a_{i'}$ and all q_{ij} for $j = 1, \ldots, n(i)$ are in the same multiplicative component $b_{i''}$. Using the branching and transposition operators (both versions, additive and multiplicative) we can construct three mixed terms $t_1 : a \to (\overline{a}(p_{11}) \otimes \cdots \otimes \overline{a}(p_{1n(1)})) \oplus \cdots \oplus (\overline{a}(p_{m1}) \otimes \cdots \otimes \overline{a}(p_{mn(m)}))$, $t_2 : (\overline{a}(p_{11}) \otimes \cdots \otimes \overline{a}(p_{1n(1)})) \oplus \cdots \oplus (\overline{a}(p_{m1}) \otimes \cdots \otimes \overline{a}(p_{mn(m)})) \to b'$, and $t_3 : b' \to b$ such that t_2 includes only combinations of additive transpositions and $f = [\![t_1]\!] \cdot [\![t_2]\!] \cdot [\![t_3]\!] = [\![t_1 \cdot t_2 \cdot t_3]\!]$. Moreover the relation $[\![t_1]\!]$ is surjective, the relation $[\![t_2]\!]$ is bijective and the relation $[\![t_3]\!]$ is total. Therefore the demonic sequential composition of the three relations is the same with the angelic sequential composition. □

Example 3. Consider $a = (a_1 \otimes a_2) \oplus (a_1 \otimes a_3)$, $b = (a_2 \otimes a_3) \oplus (a_1 \otimes a_2 \otimes a_3)$, and $f = \{(1.1 \mapsto 2.1, 1.2 \mapsto 2.2) \mid (1.2 \mapsto 1.1) \mid (2.1 \mapsto 2.1, 2.2 \mapsto 2.3) \mid (2.2 \mapsto 1.2)\}$

where $a_1, a_2, a_3 \in S$. Then $t_1 : a \to (a_1 \otimes a_2) \oplus a_2 \oplus (a_1 \otimes a_3) \oplus a_3$ is given by:

$$t_1 = ({}_{a_1 \otimes a_2}\!\bullet\!\!<_2 \oplus {}_{a_1 \otimes a_3}\!\bullet\!\!<_2) \cdot (\mathsf{I}_{a_1 \otimes a_2} \oplus (\wedge_0^{a_1} \otimes \mathsf{I}_{a_2}) \oplus \mathsf{I}_{a_1 \otimes a_3} \oplus (\wedge_0^{a_1} \otimes \mathsf{I}_{a_3})),$$

$t_2 : (a_1 \otimes a_2) \oplus a_2 \oplus (a_1 \otimes a_3) \oplus a_3 \to a_2 \oplus a_3 \oplus (a_1 \otimes a_2) \oplus (a_1 \oplus a_3)$ is given by:

$$t_2 = ({}^{a_2}_{a_1 \otimes a_2}\!\!\times \oplus {}^{a_3}_{a_1 \otimes a_3}\!\!\times) \cdot (\mathsf{I}_{a_2} \oplus {}^{a_3}_{a_1 \otimes a_2}\!\!\times \oplus \mathsf{I}_{a_1 \otimes a_3}),$$

and $t_3 : a_2 \oplus a_3 \oplus (a_1 \otimes a_2) \oplus (a_1 \oplus a_3) \to b$ is given by:

$$t_3 = ((\mathsf{I}_{a_2} \otimes \vee^0_{a_3}) \oplus (\vee^0_{a_2} \otimes \mathsf{I}_{a_3}) \oplus (\mathsf{I}_{a_1 \otimes a_2} \otimes \vee^0_{a_3}) \oplus (\mathsf{I}_{a_1} \otimes \vee^0_{a_2} \otimes \mathsf{I}_{a_3})) \cdot$$
$$({}_2\!>\!\!\bullet_{a_2 \otimes a_3} \oplus {}_2\!>\!\!\bullet_{a_1 \otimes a_2 \otimes a_3}).$$

A careful analysis of the above example points out that t_1 is of the form $t_{11} \cdot t_{12}$ where t_{11} is a sum of additive branching operators and t_{12} is a sum of products of multiplicative branching operators. A similar thing is true for t_3. It is not hard to see that this property holds for the general case. Therefore a refined version of Lemma 8 holds:

Corollary 9. *Let a and b two sorts in additive normal forms. For each f in $\mathsf{MixRel}_S(a,b)$ there exist the mixed terms $t_1, \ldots, t_n$ such that, for each i, either t_i is a sum of additive (branching and transposition) operators or t_i is a sum of products of multiplicative (branching and transposition) operators and $[\![t_1 \cdots t_n]\!] = f$.*

Theorem 10. (Expressiveness.) *Let a and b two arbitrary sorts. For each f in $\mathsf{MixRel}_S(a,b)$ there is a mixed term t such that $[\![t]\!] = f$.*

Proof. Let f' be the mixed relation of sort $\mathsf{nf}_a \to \mathsf{nf}_b$ such that $p' \mapsto q' \in f'$ iff there exist p and q such that $p \mapsto q \in f$, $p \mapsto p' \in [\![\mathsf{dis}_a]\!]$, $q \mapsto q' \in [\![\mathsf{dis}_b]\!]$. Then the mixed term required by the theorem is $t = \mathsf{dis}_a \cdot t' \cdot \mathsf{dis}_b^{-1}$ where dis_b^{-1} is the mixed term with the property $[\![\mathsf{dis}_b^{-1}]\!] = [\![\mathsf{dis}_b]\!]^{-1}$. The mixed term dis_a^{-1} exists because $[\![\mathsf{dis}_a]\!]$ is an isomorphism. Recall that the relation $[\![\mathsf{dis}\_]\!]$ preserves the maximal multiplicative worlds and therefore the demonic sequential compositions given above are just those we need to be. □

Theorem 11. (Soundness.) *Let E denote the set of axioms B1-2, AddB3-10, MultB3-10, D1-10, SV1-20, AddS1-4, and MultS1-4. If $E \vdash t_1 = t_2$ then $[\![t_1]\!] = [\![t_2]\!]$.*

Proof. It is a matter of routine to check that $[\![t_1]\!] = [\![t_2]\!]$ for each axiom $t_1 = t_2$ in E.

For example, we check the axiom AddA4. The mixed relation denoted by the left hand side is given by $[\![{}_2\!>\!\!\bullet_a \cdot {}_a\!\bullet\!\!<_0]\!] = [\![{}_2\!>\!\!\bullet_a]\!] \cdot [\![{}_a\!\bullet\!\!<_0]\!] = \{\!|\Downarrow\!1.p \mapsto p, \Downarrow\!2.p \mapsto p|\!\} \cdot \{\,\}_{a \to 0} = \{\,\}_{a \oplus a \to 0}$ and the mixed relation denoted by the right hand side is given by $[\![{}_a\!\bullet\!\!<_0 \oplus {}_a\!\bullet\!\!<_0]\!] = \{\,\}_{a \to 0} \oplus \{\,\}_{a \to 0} = \{\,\}_{a \oplus a \to 0}$. Obviously, the two relations are equal. □

Corollary 12. *The category MixRel_S whose objects are sorts over S and whose arrows are mixed relations is an enriched semiringal category. The strict subcategory $x_1x_2x_3x_4$-MixRel_S of $x_1x_2x_3x_4$-relations is a $x_1x_2x_3x_4$-enriched semiringal category.* □

Theorem 13. (Completeness) *Let M be an $x_1x_2x_3x_4$-enriched symmetric semiringal category and $h : S^{\oplus,\otimes} \to M$ a homomorphism of $(\oplus, \otimes, 0, 1)$-algebras. Then there is a unique homomorphism of $x_1x_2x_3x_4$-enriched symmetric semiringal categories $H : x_1x_2x_3x_4\text{-MixRel}_S \to M$ which extends h.*

Proof. We say a relation $f : a \to b$ is written in the *additive mixed normal form* (shortly, additive `mixnf`) if it has the following shape ($red\_$ is the inverse of $dis\_$)

$$f = dis_a \cdot g \cdot (r^1 \oplus \ldots \oplus r^k) \cdot h \cdot red_b$$

where

- g^{-1}, h are additive normal form terms which represents functions — that is, g is additive ad-term and h is additve da-term, and
- $r^1, \ldots, r^k$ are multiplicative normal form terms.

In the classifying MixNA notation of Section 1, g is a $daaa$-relation, h is an $adaa$-relations and $r^1, \ldots, r^k$ are $aadd$-relations. Moreover, in the particular case when g and h are identities and $k = 1$ the resulting relational term is a sum of multiplicative normal forms, while in the case $r^1, \ldots, r^k$ are identities, the result is an additive normal form.

Here we give the main lines of the proof of the Theorem.

We separate the axioms in

M1 $(M, \oplus, \cdot, \mathsf{I}, \mathsf{X}, \bullet\!\!<_k, {}_k\!\!>\!\!\bullet)$ fulfills the angelic additive axioms AddA1–11 in Section A;

M2 $(M, \otimes, \cdot, \mathsf{I}, \mathsf{X}, \wedge_k, \vee^k)$ fulfills the forward-demonic multiplicative axioms MultA1–11 in Section A;

M3 the scalar–vectorial axioms for the additive and multiplicative branching constants hold, namely SV1–20 in Section A.

From the enriched semiringal category axioms it follows that:

M4 the additive branching constants $\bullet\!\!<_k, {}_k\!\!>\!\!\bullet$ commute with arbitrary multiplicative terms (i.e., the additive strong axioms hold whenever f is a term over $(M, \otimes, \cdot, \mathsf{I}, \mathsf{X}, \wedge_k, \vee^k)$);

M5 $({}_a\!\bullet\!\!<_k \otimes {}_a\!\bullet\!\!<_l) \cdot \delta_{a,\overset{k}{\ldots},a;a,\overset{k}{\ldots},a} = {}_{a\otimes a}\!\bullet\!\!<_{kl}$
${}_k\!\!>\!\!\bullet_a \otimes {}_l\!\!>\!\!\bullet_a = \delta_{a,\overset{k}{\ldots},a;a,\overset{k}{\ldots},a} \cdot {}_{kl}\!\!>\!\!\bullet_{a\otimes a}$

The proof of the completness part consists in two steps:

(a) each term may be brought to an additive normal form `mixnf` using the axioms; and

(b) two additive `mixnf` forms which represent the same relation may be transformed one into the other using the axioms.

For (a), we prove that sum, product and composition of additive mixed normal forms may be brought to a normal form via the axioms.

1. $f \oplus f'$**:** If $f = dis_a \cdot g \cdot r \cdot h \cdot red_b$ and $f' = dis_{a'} \cdot g' \cdot r' \cdot h' \cdot red_{b'}$ are two additive `mixnf`s, then using the distributivity of $\oplus$ over $\cdot$ one gets $f \oplus f' = (dis_a \oplus dis_{a'}) \cdot (g \oplus g') \cdot (r \oplus r') \cdot (h \oplus h') \cdot (red_b \oplus red_{b'})$. Since $dis_a \oplus dis_{a'} = dis_{a\oplus a'}$ and $red_b \oplus red_{b'} = red_{b\oplus b'}$ the resulting term gives an additive `mixnf` for sum.

2. $f \otimes f'$**:** Suppose $f = dis_a \cdot g \cdot r \cdot h \cdot red_b$ and $f' = dis_{a'} \cdot g' \cdot r' \cdot h' \cdot red_{b'}$ are two additive `mixnf`s. Using the distributivity of $\otimes$ over $\cdot$ one gets $f \otimes f' = (dis_a \otimes dis_{a'}) \cdot (g \otimes g') \cdot (r \otimes r') \cdot (h \otimes h') \cdot (red_b \otimes red_{b'})$. Next, by the distributivity of $\otimes$ over $\oplus$, each term $g \otimes g'$, $r \otimes r'$, or $h \otimes h'$ may be written as a sum of (tensor) products. For the middle term $r \otimes r'$ it is clear that the tensor product of two multiplicative normal forms $r_i \otimes r'_j$ may be brought to a multiplicative nf by the distributivity of $\otimes$ over $\cdot$. The first (resp. last) term is a sum of additive branching constants. By the distributivity of $\otimes$ over $\oplus$, one gets sums of terms $c \otimes c'$, where c and c' are $\bullet\!\!<_k$ (resp. ${}_k\!\!>\!\!\bullet$) constants; then by axioms M5 one finally gets an additive `mixnf`. (This method applies to the bijections in g or h, as well. E.g., they may be written as a composite of sums of identities I and transpositions X.) The dis and red constants in between the factors are anichilated, while the ones from the top (resp. the bottom) contributes to the final dis (resp. red) factor.

3. $f \cdot f'$**:** Suppose $f = dis_a \cdot g \cdot r \cdot h \cdot red_b$ and $f' = dis_b \cdot g' \cdot r' \cdot h' \cdot red_c$ are two additive `mixnf`s. We apply the standard procedure of [CaS91] to normalize the composite of two additive normal forms. By M1, one may commute h and g' to get $f \cdot f' = dis_a \cdot g \cdot r \cdot a'_1(g') \cdot \phi \cdot c_1(h) \cdot r' \cdot h' \cdot red_c$ where $a'_1(g')$ (resp. $c_1(h)$) is an appropriate sum of the same type as g' (resp. h) and ϕ is an additive bijective term. Next, by the strong axioms M4, $a'_1(g')$ commutes with r (resp. $c_1(h)$ comutes with r'), hence one gets $f \cdot f' = dis_a \cdot g \cdot a'_2(g') \cdot b_1(r) \cdot \phi \cdot b'_1(r') \cdot c_2(h) \cdot h' \cdot red_c$ where $a'_2(g')$ (resp. $b_1(r), b'_1(r')$, or $c_2(h)$) is an appropriate sum of the same type as g' (resp. r, r', or h). By axioms M1 (for X) ϕ may be commuted with $b_1(r)$, say, and thereafter it may be incorporated into $a'_2(g')$. One gets a new term $f \cdot f' = dis_a \cdot [g \cdot a'_3(g')] \cdot [b_2(r) \cdot b'_1(r')] \cdot [c_2(h) \cdot h'] \cdot red_c$ where $a'_3(g')$ (resp. $b_2(r)$) is an appropriate sum of the same type as g' (resp. r). By M1 axioms the first and the last $[\ldots]$ factors may be brought to appropriate additive nf. The middle $[\ldots]$ factor is a composite of sums of multiplicative nfs, and by the distributivity of $\oplus$ over $\cdot$ it may be written as a sum of composites of multiplicative nfs. By M2 axioms each term of the sum may be brought to a multiplicative nf, hence an additive `mixnf` for $f \cdot f'$ is finally obtained.

For (b), one has to notice that two additive normal form mixed terms which represent the same mixed relation may be transformed one into the other using the standard procedure in [CaS91]. (The reduction is based on the fact that the set of multiplicative relations r_i correspond to the multiplicative worlds, hence their set is the same in both representation.) □

Corollary 14. *If* t_1 *and* t_2 *are two mixed terms such that* $[\![t_1]\!] = [\![t_2]\!]$ *then* $E \vdash t_1 = t_2$ *where* E *is defined as in Theorem 11.* □

Remark. The point of view taken in [CaS91, Ste94] is to get axiomatizations in a "linear" setting, i.e. to avoid the use of the strong axioms of type S1–S4. (These axioms allow to make copies of or to delete arbitrary morphisms.)

An interesting observation is that the additive strong axioms follow from the distributivity axiom of the semiringal categories. Indeed, using ${}_a\!\bullet\!\!<_k = ({}_1\!\bullet\!\!<_k \otimes \mathsf{I}_a) \cdot \delta_{1,\overset{k}{\ldots},1;a}$ we get

$f \cdot {}_b\!\bullet\!\!<_k$

$$= f \cdot ({}_1{\bullet\!\!<}_k \otimes \mathsf{I}_b) \cdot \delta_{1,\overset{k}{\ldots},1;b}$$
$$= (\mathsf{I}_1 \otimes f) \cdot ({}_1{\bullet\!\!<}_k \otimes \mathsf{I}_b) \cdot \delta_{1,\overset{k}{\ldots},1;b}$$
$$= ({}_1{\bullet\!\!<}_k \otimes \mathsf{I}_a) \cdot (\mathsf{I}_k \otimes f) \cdot \delta_{1,\overset{k}{\ldots},1;b}$$
$$= ({}_1{\bullet\!\!<}_k \otimes \mathsf{I}_a) \cdot ((\mathsf{I}_1 \oplus \overset{k}{\ldots} \oplus \mathsf{I}_1) \otimes f) \cdot \delta_{1,\overset{k}{\ldots},1;b}$$
$$= ({}_1{\bullet\!\!<}_k \otimes \mathsf{I}_a) \cdot \delta_{1,\overset{k}{\ldots},1;a} \cdot (\mathsf{I}_1 \otimes f \oplus \overset{k}{\ldots} \oplus \mathsf{I}_1 \otimes f)$$
$$= {}_a{\bullet\!\!<}_k \cdot (f \oplus \overset{k}{\ldots} \oplus f)$$

(A similar fact holds for ${>\!\!\bullet}$.)

For this reason, we had to weaken the semiringal category structure by requiring that the distributivity axioms hold for particular classes of morphisms, only. An $x_1x_2x_3x_4$-weak semiringal category is one where the distributivity axiom is required for morphisms which are represented by $x_1x_2x_3x_4$-terms, only.

5 Conclusions

We have described some axiomatization results for the mixed relations used to model interface-changing relations in parallel programs. The algebraic structures involved are enriched weak symmetric semiringal categories. The option here was to use an angelic-additive forward-demonic-multiplicative version, which was induced by the standard set-theoretic semantics of MixNA.

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A Axioms

A.1 Symmetric semiringal category axioms

B1 $f \cdot (g \cdot h) = (f \cdot g) \cdot h$
B2 $\mathsf{I}_a \cdot f = f = f \cdot \mathsf{I}_b$

AddB3 $f \oplus (g \oplus h) = (f \oplus g) \oplus h$
AddB4 $\mathsf{I}_0 \oplus f = f = f \oplus \mathsf{I}_0$
AddB5 $(f \oplus f') \cdot (g \oplus g') = f \cdot g \oplus f' \cdot g'$
AddB6 $\mathsf{I}_a \oplus \mathsf{I}_b = \mathsf{I}_{a\oplus b}$
AddB7 ${}^b_a\mathsf{X} \cdot {}^a_b\mathsf{X} = \mathsf{I}_{a\oplus b}$
AddB8 ${}^0_a\mathsf{X} = \mathsf{I}_a$
AddB9 ${}^{b\oplus c}_{a}\mathsf{X} = ({}^b_a\mathsf{X} \oplus \mathsf{I}_c) \cdot (\mathsf{I}_b \oplus {}^c_a\mathsf{X})$
AddB10 $(f \oplus g) \cdot {}^d_c\mathsf{X} = {}^b_a\mathsf{X} \cdot (g \oplus f)$ for $f : a \to c, g : b \to d$

MultB3-MultB10 denote the axioms obtained from AddB3-AddB10 by replacing the additive operators with the corresponding multiplicative ones.

Below we will use the derivated constants $\delta_{a,b;c}$ and $\delta_{;c}$ defined by the following rules:
– $\delta_{a,b;c} = {}^{a\oplus b}\mathsf{X}^{c} \cdot \delta_{c;a,b} \cdot ({}^{c}\mathsf{X}^{a} \oplus {}^{c}\mathsf{X}^{b})$
– $\delta_{;c} = {}^{0}\mathsf{X}^{c} \cdot \delta_{c;}$

D1 $\delta_{a;0,d} = \mathsf{I}_{a\otimes d}$
D2 $\delta_{a;b\oplus c,d} \cdot (\delta_{a;b,c} \oplus \mathsf{I}_{a\otimes d}) = \delta_{a;b,c\oplus d} \cdot (\mathsf{I}_{a\otimes b} \oplus \delta_{a;c,d})$
D3 $\delta_{a;b,c} \cdot {}^{a\otimes c}_{a\otimes b}\mathsf{X} = (\mathsf{I}_{a} \otimes {}^{c}_{b}\mathsf{X}) \cdot \delta_{a;c,b}$
D4 $\delta_{1;b,c} = \mathsf{I}_{b\oplus c}$
D5 $\delta_{a'\otimes a'';b,c} = (\mathsf{I}_{a'} \otimes \delta_{a'';b,c}) \cdot \delta_{a';a''\otimes b,a''\otimes c}$
D6 $\delta_{a'\oplus a'';b,c} \cdot (\delta_{a',a'';b} \oplus \delta_{a',a'';c}) = \delta_{a',a'';b\oplus c} \cdot (\delta_{a';b,c} \oplus \delta_{a'';b,c}) \cdot (\mathsf{I}_{a'\otimes b} \oplus {}^{a''\otimes b}_{a'\otimes c}\mathsf{X} \oplus \mathsf{I}_{a''\otimes c})$
D7 $\delta_{1;} = \mathsf{I}_{0}$
D8 $\delta_{a'\otimes a'';} = (\mathsf{I}_{a'} \otimes \delta_{a'';}) \cdot \delta_{a';}$
D9 $\delta_{a'\oplus a'';} = \delta_{a',a'';0} \cdot (\delta_{a;} \oplus \delta_{a'';})$
D10 $(f \otimes (g \oplus h)) \cdot \delta_{a';b',c'} = \delta_{a;b,c} \cdot ((f \otimes g) \oplus (f \otimes h))$
D11 $(f \otimes \mathsf{I}_{0}) \cdot \delta_{a';} = \delta_{a;}$

A.2 Additional axioms for angelic additive *dd*-ssmc

AddA1 $({}_{2}{\succ\!\bullet}_{a} \oplus \mathsf{I}_{a}) \cdot {}_{2}{\succ\!\bullet}_{a} = (\mathsf{I}_{a} \oplus {}_{2}{\succ\!\bullet}_{a}) \cdot {}_{2}{\succ\!\bullet}_{a}$
AddA2 ${}^{a}_{a}\mathsf{X} \cdot {}_{2}{\succ\!\bullet}_{a} = {}_{2}{\succ\!\bullet}_{a}$
AddA3 $({}_{0}{\succ\!\bullet}_{a} \oplus \mathsf{I}_{a}) \cdot {}_{2}{\succ\!\bullet}_{a} = \mathsf{I}_{a}$
AddA4 ${}_{2}{\succ\!\bullet}_{a} \cdot {}_{a}{\bullet\!\prec}_{0} = {}_{a}{\bullet\!\prec}_{0} \oplus {}_{a}{\bullet\!\prec}_{0}$
AddA5 ${}_{a}{\bullet\!\prec}_{2} \cdot ({}_{a}{\bullet\!\prec}_{2} \oplus \mathsf{I}_{a}) = {}_{a}{\bullet\!\prec}_{2} \cdot (\mathsf{I}_{a} \oplus {}_{a}{\bullet\!\prec}_{2})$
AddA6 ${}_{a}{\bullet\!\prec}_{2} \cdot {}^{a}_{a}\mathsf{X} = {}_{a}{\bullet\!\prec}_{2}$
AddA7 ${}_{a}{\bullet\!\prec}_{2} \cdot ({}_{a}{\bullet\!\prec}_{0} \oplus \mathsf{I}_{a}) = \mathsf{I}_{a}$
AddA8 ${}_{0}{\succ\!\bullet}_{a} \cdot {}_{a}{\bullet\!\prec}_{2} = {}_{0}{\succ\!\bullet}_{a} \oplus {}_{0}{\succ\!\bullet}_{a}$
AddA9 ${}_{0}{\succ\!\bullet}_{a} \cdot {}_{a}{\bullet\!\prec}_{0} = \mathsf{I}_{0}$
AddA10 ${}_{2}{\succ\!\bullet}_{a} \cdot {}_{a}{\bullet\!\prec}_{2} = ({}_{a}{\bullet\!\prec}_{2} \oplus {}_{a}{\bullet\!\prec}_{2}) \cdot (\mathsf{I}_{a} \oplus {}^{a}_{a}\mathsf{X} \oplus \mathsf{I}_{a}) \cdot ({}_{2}{\succ\!\bullet}_{a} \oplus {}_{2}{\succ\!\bullet}_{a})$
AddA11 ${}_{a}{\bullet\!\prec}_{2} \cdot {}_{2}{\succ\!\bullet}_{a} = \mathsf{I}_{a}$

A.3 Additional axioms for forward-demonic multiplicative *dd*-ssmc

The axioms are similar to the ones in the additive case. The only difference is in the case of axiom A3.

MultA1 $(\curlyvee^{2}_{a} \otimes \mathsf{I}_{a}) \cdot \curlyvee^{2}_{a} = (\mathsf{I}_{a} \otimes \curlyvee^{2}_{a}) \cdot \curlyvee^{2}_{a}$
MultA2 ${}^{a}\mathsf{X}^{a} \cdot \curlyvee^{2}_{a} = \curlyvee^{2}_{a}$
MultA3 $(\curlyvee^{0}_{a} \otimes \mathsf{I}_{a}) \cdot \curlyvee^{2}_{a} = \curlywedge^{a}_{0} \cdot \curlyvee^{0}_{a}$
MultA4 $\curlyvee^{2}_{a} \cdot \curlywedge^{a}_{0} = \curlywedge^{a}_{0} \otimes \curlywedge^{a}_{0}$
MultA5 $\curlywedge^{a}_{2} \cdot (\curlywedge^{a}_{2} \otimes \mathsf{I}_{a}) = \curlywedge^{a}_{2} \cdot (\mathsf{I}_{a} \otimes \curlywedge^{a}_{2})$
MultA6 $\curlywedge^{a}_{2} \cdot {}^{a}\mathsf{X}^{a} = \curlywedge^{a}_{2}$
MultA7 $\curlywedge^{a}_{2} \cdot (\curlywedge^{a}_{0} \otimes \mathsf{I}_{a}) = \mathsf{I}_{a}$
MultA8 $\curlyvee^{0}_{a} \cdot \curlywedge^{a}_{2} = \curlyvee^{0}_{a} \otimes \curlyvee^{0}_{a}$
MultA9 $\curlyvee^{0}_{a} \cdot \curlywedge^{a}_{0} = \mathsf{I}_{0}$
MultA10 $\curlyvee^{2}_{a} \cdot \curlywedge^{a}_{2} = (\curlywedge^{a}_{2} \otimes \curlywedge^{a}_{2}) \cdot (\mathsf{I}_{a} \otimes {}^{a}\mathsf{X}^{a} \otimes \mathsf{I}_{a}) \cdot (\curlyvee^{2}_{a} \otimes \curlyvee^{2}_{a})$
MultA11 $\curlywedge^{a}_{2} \cdot \curlyvee^{2}_{a} = \mathsf{I}_{a}$

A.4 Scalar vectorial axioms:

Multiplicative branching constants:

SV1 $\curlywedge_2^0 = \mathsf{I}_0$
SV2 $\curlywedge_2^1 = \mathsf{I}_1$
SV3 $\curlywedge_2^{a\oplus b} = (\curlywedge_2^a \oplus \curlywedge_2^b) \cdot (\mathsf{I}_{a\otimes a} \oplus {}_0{\succ\!\bullet}_{a\otimes b} \oplus {}_0{\succ\!\bullet}_{b\otimes a} \oplus \mathsf{I}_{b\otimes b}) \cdot \rho_{a,b;a,b}$
SV4 $\curlywedge_2^{a\otimes b} = (\curlywedge_2^a \otimes \curlywedge_2^b) \cdot (\mathsf{I}_a \otimes {}^a\mathsf{X}^b \otimes \mathsf{I}_b)$

SV5 $\curlywedge_0^0 = {}_0{\succ\!\bullet}_1$
SV6 $\curlywedge_0^1 = \mathsf{I}_1$
SV7 $\curlywedge_0^{a\oplus b} = (\curlywedge_0^a \oplus \curlywedge_0^b) \cdot {}_2{\succ\!\bullet}_1$
SV8 $\curlywedge_0^{a\otimes b} = \curlywedge_0^a \otimes \curlywedge_0^b$

SV9 $\curlyvee_0^2 = \mathsf{I}_0$
SV10 $\curlyvee_1^2 = \mathsf{I}_1$
SV11* $\curlyvee_{a\oplus b}^2 = \delta_{a,b;a,b} \cdot (\mathsf{I}_{a\otimes a} \oplus {}_0{\succ\!\bullet}_{a\otimes b} \oplus {}_0{\succ\!\bullet}_{b\otimes a} \oplus \mathsf{I}_{b\otimes b}) \cdot (\curlyvee_a^2 \oplus \curlyvee_b^2)$
SV12 $\curlyvee_{a\otimes b}^2 = (\mathsf{I}_a \otimes {}^a\mathsf{X}^b \otimes \mathsf{I}_b) \cdot (\curlyvee_a^2 \otimes \curlyvee_b^2)$

SV13 $\curlyvee_0^0 = {}_1{\bullet\!\prec}_0$
SV14 $\curlyvee_1^0 = \mathsf{I}_1$
SV15 $\curlyvee_{a\oplus b}^0 = {}_1{\bullet\!\prec}_2 \cdot (\curlyvee_a^0 \oplus \curlyvee_b^0)$
SV16 $\curlyvee_{a\otimes b}^0 = \curlyvee_a^0 \otimes \curlyvee_b^0$

Additive branching constants: Their rules follows from the scalar-vectorial rules for δ and

SV17 ${}_a{\bullet\!\prec}_2 = ({}_1{\bullet\!\prec}_2 \otimes \mathsf{I}_a) \cdot \delta_{1,1;a}$
SV18 ${}_a{\bullet\!\prec}_0 = ({}_1{\bullet\!\prec}_0 \otimes \mathsf{I}_a) \cdot \delta_{;a}$
SV19 ${}_2{\succ\!\bullet}_a = \rho_{1,1;a} \cdot ({}_2{\succ\!\bullet}_1 \otimes \mathsf{I}_a)$
SV20 ${}_0{\succ\!\bullet}_a = \rho_{;a} \cdot ({}_0{\succ\!\bullet}_1 \otimes \mathsf{I}_a)$

Note: It is a problem with the meaning of the equality test in the case the terms to be compared are of arbitrary type and not simple elements, or tuples of elements. The choice SV11* above may look well, but is not valid in certain natural semantics models $MixRel_S(D)$.

A.5 The strong commutativity axioms for branching constants

AddS1 ${}_0{\succ\!\bullet}_a \cdot f = {}_0{\succ\!\bullet}_b$
AddS2 ${}_2{\succ\!\bullet}_a \cdot f = (f \oplus f) \cdot {}_2{\succ\!\bullet}_b$
AddS3 $f \cdot {}_b{\bullet\!\prec}_0 = {}_a{\bullet\!\prec}_0$
AddS4 $f \cdot {}_b{\bullet\!\prec}_2 = {}_a{\bullet\!\prec}_2 \cdot (f \oplus f)$

MultS1 $\curlyvee_a^0 \cdot f = \curlyvee_b^0$
MultS2 $\curlyvee_a^2 \cdot f = (f \otimes f) \cdot \curlyvee_b^2$
MultS3 $f \cdot \curlywedge_0^b = \curlywedge_0^a$
MultS4 $f \cdot \curlywedge_2^b = \curlywedge_2^a \cdot (f \otimes f)$

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