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An Effective Tietze-Urysohn Theorem for QCB-Spaces

Matthias Schröder
(Universität der Bundeswehr, Munich, Germany
matthias.schroeder@unibw.de)

Abstract: The Tietze-Urysohn Theorem states that every continuous real-valued function defined on a closed subspace of a normal space can be extended to a continuous function on the whole space. We prove an effective version of this theorem in the Type Two Model of Effectivity (TTE). Moreover, we introduce for qcb-spaces a slightly weaker notion of normality than the classical one and show that this property suffices to establish an Extension Theorem for continuous functions defined on functionally closed subspaces. Qcb-spaces are known to form an important subcategory of the category Top of topological spaces. QCB is cartesian closed in contrast to Top.
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Category: F.1.1

1 Introduction

Theorems about extendability of continuous functions belong to the most important theorems in the field of topological spaces. Extendability of a continuous function f onto a larger space Y means the existence of a continuous function F on Y which coincides with f on the domain of f. A famous example of an extension theorem is the Tietze-Urysohn Theorem for normal topological spaces. Of similar interest are theorems about extendability of *computable* functions. A computable version of the Tietze-Urysohn Theorem for computable metric spaces has been proved by K. Weihrauch in [Weihrauch 01].

In this paper we prove a continuous and a computable Extension Theorem for a subclass of qcb-spaces that contains all computable metric spaces. Qcb-spaces [Simpson 03] are known to form exactly the class of topological spaces which can be handled by the representation based approach to Computable Analysis, the Type Two Model of Effectivity (TTE). The category QCB of qcb-spaces has excellent closure properties, for example it is cartesian closed [Schröder 03, Simpson et al. 07].

Unfortunately, many interesting Hausdorff qcb-spaces fail to be normal. For example, it was recently proved that the space $\mathbb{N}^{(\mathbb{N}^{\mathbb{N}})}$ of Kleene-Kreisel continuous functionals of order 2 is not regular [Schröder 09]. Moreover, the space of real-valued continuous functions on a computable metric space need not necessarily be normal in QCB (cf. [Schröder 09]). Hence the classical Tietze-Urysohn Theorem, which requires normality, can not be applied to these kinds of spaces.

In this paper we introduce a weaker notion of normality called *quasi-normality*. This notion may be considered as a substitute for normality in the class of qcb-spaces (cf. Section 3). We show that quasi-normal qcb-spaces admits extendability of continuous functions defined on functionally closed subspaces (cf. Section 4). The category QN of quasi-normal qcb-spaces forms a subcategory of QCB that contains all separable metrisable spaces and inherits the cartesian closed structure of QCB by being an exponential ideal of QCB.

In Section 5 we establish a computable version of the Tietze-Urysohn Theorem. It is formulated for qcb-spaces that satisfy an effective notion of quasi-normality.

2 Preliminaries

After fixing some notations, we repeat some notions and basic facts of topological spaces, of the used computational model, of qcb-spaces and of pseudobases.

2.1 Notations

We write $\mathbb{N}$ for the set of natural numbers (including 0) and also for the discrete topological space with carrier set $\mathbb{N}$. The set of infinite sequences over $\mathbb{N}$ is denoted by $\mathbb{N}^{\mathbb{N}}$, the set of finite words over $\mathbb{N}$ by $\mathbb{N}^*$, and, for a word $w \in \mathbb{N}^*$, the set of sequences with prefix w by $w\mathbb{N}^{\mathbb{N}}$. We write $p^{<k}$ for the prefix of $p \in \mathbb{N}^{\mathbb{N}}$ of length k and $\sqsubseteq$ for the prefix relation on $\mathbb{N}^* \cup \mathbb{N}^{\mathbb{N}}$.

Depending on the context, $\langle . \rangle$ stands for a computable bijection either from $(\mathbb{N}^{\mathbb{N}})^k$ to $\mathbb{N}^{\mathbb{N}}$ or from $(\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}$ to $\mathbb{N}^{\mathbb{N}}$ or from $\mathbb{N}^k$ to $\mathbb{N}$, as defined in [Schröder 03]. Moreover, we denote by $\mathsf{w}\colon \mathbb{N} \to \mathbb{N}^*$ an effective bijection between $\mathbb{N}$ and $\mathbb{N}^*$. For a subset $M \subseteq \mathbb{R}$, ϱ_M stands for the binary signed-digit representation corestricted to M.

2.2 Computability theory

As the underlying computational model we use the representation-based approach to Computable Analysis, the Type-2 Theory of Effectivity (TTE). We assume that the reader is familiar with basic concepts of TTE, see [Weihrauch 00, Weihrauch 08].

We only repeat here the less known notion of a computable multi-function. A *multi-function* (or *operator*) $\varPhi$ from X to Y is a relation between X and Y. The domain of $\varPhi$ is the set $\mathrm{dom}(\varPhi) := \{x \in X \mid \exists y. (x,y) \in \varPhi\}$. Given two representations $\delta :\subseteq \mathbb{N}^{\mathbb{N}} \to X$ of X and $\gamma :\subseteq \mathbb{N}^{\mathbb{N}} \to Y$ of Y, a multi-function $\varPhi\colon X \rightrightarrows Y$ is called *computable* (w.r.t. δ and γ), if there is partial computable function $g :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ which maps any name p of an element $x \in \mathrm{dom}(\varPhi)$ to a name of *one* possible result for x, i.e. for all $p \in \mathrm{dom}(\delta)$ with $\delta(p) \in \mathrm{dom}(\varPhi)$ we have $\gamma(g(p)) \in \{y \in Y \mid (x,y) \in \varPhi\}$. Computability of ordinary functions is defined correspondingly.

2.3 Topological spaces and sequential spaces

Our references to the theory of topological spaces are [Engelking 89, Willard 70]. To denote topological spaces, we use sans-serif letters like X, Y etc. We write $\mathcal{O}(\mathsf{X})$ for the topology of a space X, $\mathcal{A}(\mathsf{X})$ for the family of closed sets of X and $\mathcal{G}(\mathsf{X})$ for the family of $\mathcal{G}_\delta$-sets of X, which are countable intersections of open sets. By abuse of notation, we will often denote the carrier set of a space X by the same symbol X.

A subset A of a topological space X is called *sequentially closed*, if A contains any limit of any convergent sequence of elements in A. Complements of sequentially closed sets are called *sequentially open*. For a given topology τ, we denote the topology of sequentially open sets by $seq(\tau)$. Spaces such that every sequentially open set is open are called *sequential*. The sequential coreflection (or sequentialisation) $seq(\mathsf{X})$ of X is the topological space that carries the topology $seq(\mathcal{O}(\mathsf{X}))$ of sequentially open sets of X. The operator seq is idempotent.

A subset A of a topological space X is called *functionally closed*, if there is a continuous function f from X to the unit interval $\mathbb{I} = [0, 1]$ (endowed with the usual Euclidean topology) such that $f^{-1}\{0\} = A$. Complements of functionally closed sets are called *functionally open*. A common term for "functionally closed set" is *zero-set*. Two disjoint functionally closed sets A, B can be strongly separated in the sense that there is a continuous function $h\colon \mathsf{X} \to [0, 1]$ satisfying $h^{-1}\{0\} = A$ and $h^{-1}\{1\} = B$.

We denote the family of functionally open sets of X by $\mathcal{FO}(\mathsf{X})$ and the family of functionally closed sets by $\mathcal{FA}(\mathsf{X})$. T_0-spaces such that all open sets are functionally open are called *perfectly normal*. Functionally open sets are closed under finite union, but not necessarily under arbitrary union, unless X is hereditarily Lindelöf space (i.e. any open cover of any subset has a countable subcover). In this case $\mathcal{FO}(\mathsf{X})$ forms a topology. It has the property that every real-valued function f on X is continuous w.r.t. the original topology $\mathcal{O}(\mathsf{X})$ if, and only if, f is continuous w.r.t. $\mathcal{FO}(\mathsf{X})$. Regularity, normality and perfect normality are equivalent for hereditarily Lindelöf spaces and thus for qcb-spaces, which are defined below. For us, a *normal* space is a T_1-space (some authors omit the T_1-condition) such that for a pair (A, B) of disjoint closed sets there exists a pair (U, V) of disjoint open sets such that $A \subseteq U$ and $B \subseteq V$.

2.4 Qcb-spaces and admissible representations

A qcb-space [Simpson 03] is a topological *q*uotient of a *c*ountably-*b*ased topological space. Qcb$_0$-spaces, i.e. qcb-spaces that satisfy the T_0-property, are exactly the class of sequential spaces which have an *admissible* representation and which can therefore be handled by the Type Two Model of Effectivity. Admissibility is a property guaranteeing topological well-behavedness of representations (cf.

[Schröder 02]). The final topology of an admissible representation of a sequential space is equal to the topology of that space.

Qcb-spaces are hereditarily Lindelöf and sequential. The category QCB of qcb-spaces as objects and of continuous functions as morphisms is cartesian closed. Moreover, QCB has all countable limits and all countable colimits. For two admissible representations δ_{X} and δ_{Y} of qcb$_0$-spaces X and Y we denote by $[\delta_{\mathsf{X}} \to \delta_{\mathsf{Y}}]$ the usual admissible function space representation of Y^{X} as defined in [Schröder 03] or [Weihrauch 00].

More information can be found in [Schröder 02, Schröder 03, Simpson 03, Simpson et al. 07].

2.5 Pseudobases and pseudo-open decompositions

Given a topological space X, we say that a family $\mathcal{A}$ of subsets of X is a *pseudo-open decomposition* of a subset M, if $M = \bigcup \mathcal{A}$ holds and for every sequence $(x_n)_n$ that converges to some element $x_\infty \in M$ there is some set $B \in \mathcal{A}$ and some $n_0 \in \mathbb{N}$ such that $\{x_n, x_\infty \,|\, n \geq n_0\} \subseteq B \subseteq M$ holds. Clearly, a set has a pseudo-open decomposition if, and only if, it is sequentially open.

A *pseudobase* for X is a family $\mathcal{B}$ of subsets such that every open set has a pseudo-open decomposition into sets in $\mathcal{B}$. Any base of topological space is a pseudobase, but not vice versa. Pseudobases are of interest, when they are countable: Every admissible representation δ of a topological space X induces a countable pseudobase for X, namely the family $\mathcal{B}_\delta := \{\emptyset, \delta(w\mathbb{N}^\omega) \,\big|\, w \in \mathbb{N}^*\}$. Using the bijection $\mathsf{w}\colon \mathbb{N} \to \mathbb{N}^*$ from Section 2.1, we equip $\mathcal{B}_\delta$ with a numbering B_δ defined by $B_\delta(0) := \emptyset$ and $B_\delta(i+1) := \delta(\mathsf{w}(i)\mathbb{N}^{\mathbb{N}})$. Conversely, if $\mathcal{A}$ is a pseudobase of a sequential T_0-space, then the space has an admissible representation such that the induced pseudobase is equal to the closure of $\mathcal{A} \cup \{\emptyset\}$ under finite intersection. Hence a sequential T_0-space is a qcb-space if, and only if, it has a countable pseudobase.

3 Quasi-normal Qcb-Spaces

In this section we introduce and investigate the notion of a quasi-normal qcb-space.

The classical Tietze-Urysohn Theorem is formulated for normal spaces. Unfortunately, many interesting Hausdorff qcb-spaces fail to be normal. For example, a recent result states that the function space $\mathbb{N}^{(\mathbb{N}^\mathbb{N})}$ formed in the category QCB is not normal [Schröder 09]. Hence the final topology of the natural representation on $\mathbb{N}^{(\mathbb{N}^\mathbb{N})}$ is not normal, because it is equal to the topology of the qcb-space $\mathbb{N}^{(\mathbb{N}^\mathbb{N})}$. Moreover, the space $\mathbb{R}^{(\mathbb{R}^\mathbb{R})}$ of continuous real-valued function from $\mathbb{R}^\mathbb{R}$ to $\mathbb{R}$ is not normal either, despite the fact that the compact-open topology on $\mathbb{R}^{(\mathbb{R}^\mathbb{R})}$ is normal and the sequential coreflection of the latter yields the topology of $\mathbb{R}^{(\mathbb{R}^{\mathbb{R}})}$. Therefore we need an appropriate substitute for the property of normality.

3.1 Definition of quasi-normal qcb-spaces

The idea behind the definition of quasi-normality is the fact that finite products and function spaces in the category QCB are constructed as the sequential coreflection (sequentialisation) of their counterparts in classical topology. These preserve regularity and even normality in the case of countably pseudobased spaces.

Definition 1. A qcb-space X is called *quasi-normal*, if X is the sequential coreflection of a normal space.

In other words, a qcb-space is quasi-normal if, and only if, its convergence relation is induced by a normal topology. Clearly, a quasi-normal space is Hausdorff. Simple examples of quasi-normal spaces are normal qcb-spaces, as qcb-spaces are equal to their sequential coreflection. Quasi-normality does not imply normality. E. Michael gave an example of a normal space such that its sequential coreflection is not regular (see Example 1.2 in [Michael 73]). This sequential coreflection turns out to have a countable pseudobase, hence it is a qcb-space.

3.2 Characterisations of quasi-normality

We will now give several characterisations of quasi-normality.

The first characterisations follow from the fact that a space is hereditarily Lindelöf, if its sequential coreflection is a qcb-space.

Proposition 2. *A qcb-space* X *is quasi-normal if, and only if, it is the sequential coreflection of a regular space. A qcb-space* X *is quasi-normal if, and only if, it is the sequential coreflection of a perfectly normal space.*

Proof. Regularity, normality and perfect normality coincide in the realm of hereditarily Lindelöf spaces and thus in the realm of spaces for which the sequential coreflection is a qcb-space. □

Recall that the family $\mathcal{FO}(\mathsf{X})$ of functionally open sets of a qcb-space X forms a topology (see Section 2.3).

Proposition 3. *A* qcb_0*-space* X *is quasi-normal if, and only if, its convergence relation is induced by the topology of functionally open sets of* X.

Proof. Let X be a qcb$_0$-space and X' be a normal space with $seq(\mathsf{X}') = \mathsf{X}$. Since $\mathcal{O}(\mathsf{X}') \subseteq \mathcal{O}(\mathsf{X})$ and X is hereditarily Lindelöf, X' is hereditarily Lindelöf as well and therefore perfectly normal. Thus $\mathcal{O}(\mathsf{X}') \subseteq \mathcal{FO}(\mathsf{X}) \subseteq \mathcal{O}(\mathsf{X})$. As $\mathcal{O}(\mathsf{X}')$ induces the convergence relation of X, so does $\mathcal{FO}(\mathsf{X})$.

Conversely, if $\mathcal{FO}(\mathsf{X})$ induces the convergence relation of X then X is the sequential coreflection of the space that carries the perfectly normal topology $\mathcal{FO}(\mathsf{X})$. □

Now we characterise quasi-normal qcb-spaces in terms of properties of pseudobases. Recall that qcb-spaces are known to be those sequential spaces that have a countable pseudobase (cf. Section 2.5).

Proposition 4. *A qcb-space X is quasi-normal if, and only if, it is a T_0-space and has a countable pseudobase consisting of functionally closed sets.*

The proof is based on the following surprising lemma. By a *functional $\mathcal{G}_\delta$-set* we mean a set that is a countable intersection of functionally open sets.

Lemma 5. *Let X be a qcb-space equipped with a countable pseudobase consisting of functionally closed sets. Then every open functional $\mathcal{G}_\delta$-set $V \subseteq \mathsf{X}$ is functionally open.*

Proof. Let $G_0, G_1, \ldots$ be a sequence of functionally open sets such that $V := \bigcap_{j=0}^{\infty} G_j$ is open. Let $(\beta_i)_i$ be a pseudo-open decomposition of V (see Section 2.5) into pseudobase sets. Since the functionally closed sets $\bigcup_{i=0}^{n} \beta_i$ and $\mathsf{X} \setminus G_n$ are disjoint, there exists a continuous function $h_n \colon \mathsf{X} \to [0,1]$ with $h_n^{-1}\{0\} = \mathsf{X} \setminus G_n$ and $h_n^{-1}\{1\} = \bigcup_{i=0}^{n} \beta_i$. We define a function $f \colon \mathsf{X} \to [0,1]$ by $f(x) := \inf_{n \in \mathbb{N}} h_n(x)$ and show that f is sequentially continuous with $f^{-1}\{0\} = \mathsf{X} \setminus V$. Let $(x_n)_n$ be a sequence converging in X to some x_∞.

(1) Let $x_\infty \in V$. Then there is some $i_0, n_0 \in \mathbb{N}$ such that $\{x_n | \, n \geq n_0\} \subseteq \beta_{i_0}$. Thus for all $j \geq i_0$ and $n \geq n_0$ (including $n = \infty$) we have $h_j(x_n) = 1$ and $f(x_n) = \min\{h_0(x_n), \ldots, h_{i_0}(x_n)\}$. This implies that $(f(x_n))_n$ converges to $f(x_\infty)$. Moreover, since $h_j(x_\infty) \neq 0$ for all $j \leq i_0$, $f(x_\infty) \neq 0$.

(2) Let $x_\infty \notin V$. Then there is some $j \in \mathbb{N}$ with $x_\infty \notin G_j$, hence $f(x_\infty) = h_j(x_\infty) = 0$. As $(h_j(x_n))_n$ converges to 0, $(f(x_n))_n$ converges to 0 as well.

Hence f is sequentially continuous and therefore (topologically) continuous, because X is sequential. So f is a witness for V being functionally open. □

Now we are ready to prove Proposition 4.

Proof. (Proposition 4) We denote by X' the space carrying the topology $\mathcal{FO}(\mathsf{X})$.

First, let X be a quasi-normal qcb-space. Then X has some countable pseudobase $\mathcal{B}$ and we have $seq(\mathsf{X}') = \mathsf{X}$ by Proposition 3. We define $\mathcal{B}'$ to be the family of the closures of all sets in $\mathcal{B}$ formed in the perfectly normal space X'. So $\mathcal{B}'$ consists of functionally closed sets. We show that $\mathcal{B}'$ is a pseudobase of X'.

Let U be functionally open and let $(x_n)_n$ be a sequence converging to some element $x_\infty \in U$. By regularity of X', there are a functionally open set V, a functionally closed set A and a pseudobase element $B \in \mathcal{B}$ with $x_\infty \in B \subseteq V \subseteq A \subseteq U$ and $x_n \in B$ for almost all $n \in \mathbb{N}$. The closure of B formed in X' is a subset of A and hence of U. Thus $\mathcal{B}'$ is a pseudobase of X'. The family of all finite intersections consists of functionally closed sets and forms a countable pseudobase for X by Lemma 10 in [Schröder 02].

Conversely let X be a T_0-space with a countable pseudobase consisting of functionally closed sets. Let $x \in \mathsf{X}$. We define the set A to be the countable intersection of all pseudobase sets that contain x and show $A = \{x\}$. Since X is a T_0-space, for any $y \neq x$ there is an open set V such that either $x \in V \not\ni y$ or $x \notin V \ni y$. In the first case there exists a pseudobase set B with $x \in B \subseteq V$, hence $y \notin A$. In the other case, there are, as the complements of the pseudobase sets are open, pseudobase sets C, D with $y \in C \subseteq V$ and $x \in D \subseteq \mathsf{X} \setminus C \subseteq \mathsf{X} \setminus \{y\}$. We conclude $A = \{x\}$. Hence $\{x\}$ is functionally closed and X is a T_1-space.

In order to prove $seq(\mathsf{X}') = \mathsf{X}$, it suffices to show that X and X' induce the same convergence relation for sequences. Let $(a_n)_n$ be a sequence that does not converge in X to x. Then $(a_n)_n$ contains a subsequence $(b_n)_n$ such that no subsequence of $(b_n)_n$ converges in X to x and x does not occur in the sequence $(b_n)_n$. We consider two cases:

(1) Assume that $(b_n)_n$ has a subsequence $(c_n)_n$ that converges in X to some point y. Since $\{x\}$ and $\{y\}$ are disjoint functionally closed sets, there are two disjoint functionally open sets U and V with $x \in U$ and $y \in V$. As $(c_n)_n$ is eventually in $\mathsf{X} \setminus U$, there are infinitely many n with $b_n \notin U$. Therefore neither $(b_n)_n$ nor $(a_n)_n$ converges to x in X'.

(2) Now assume that $(b_n)_n$ has no subsequence that converges in the space X. Along with the fact that in X every singleton is closed, this implies that the set $A := \{b_n \,|\, n \in \mathbb{N}\}$ is sequentially closed. Hence A is closed in X, because X is sequential. Thus $V := X \setminus A = \bigcap_{n=0}^{\infty}(\mathsf{X} \setminus \{b_n\})$ is an open functional $\mathcal{G}_\delta$-set. Lemma 5 implies that V is functionally open. Since V contains x, but no element of $(b_n)_n$, neither $(b_n)_n$ nor $(a_n)_n$ converge to x in X'.

Conversely, convergence in X implies convergence in X', because $\mathcal{FO}(\mathsf{X}) \subseteq \mathcal{O}(\mathsf{X})$. Hence X and X' induce the same convergence relation for sequences, implying $seq(\mathsf{X}') = \mathsf{X}$. We conclude that X is quasi-normal. $\square$

3.3 Constructing quasi-normal spaces

The category QN of quasi-normal qcb-spaces enjoys excellent closure properties. Indeed, quasi-normality is preserved by forming (i) countable products, (ii) subspaces, (iii) countable coproducts, and (iv) function spaces in the category of qcb-spaces. So QN inherits the cartesian closed structure of QCB. In fact, QN is an exponential ideal of QCB.

Theorem 6. *The category* QN *of quasi-normal qcb-spaces is cartesian closed. Moreover, it has all countable limits and all countable colimits.*

Proof. For a quasi-normal qcb-space X, we denote by X' the topological space endowed with the topology of functionally open sets of X.

Countable Products: Let $(\mathsf{X}_i)_i$ be a sequence of quasi-normal qcb-spaces. The countable product of these spaces formed in QCB is given by the sequential coreflection of their Tychonoff product and thus also by the sequential coreflection of the Tychonoff product of the regular spaces X_i', because both products induce the same convergence relation. The latter product is regular (cf. [Engelking 89]) and thus normal by having a countable pseudobase (cf. [Schröder 03]). Hence the countable product of $(\mathsf{X}_i)_i$ formed in QCB is quasi-normal.

Subspaces: Subspaces in qcb-spaces are known to be topologised by the sequentialisation of the subspace topology. Let X be a qcb-subspace of a quasi-normal qcb-space Y. Since the subspace topology induced by Y' on the carrier set of X is perfectly normal and induces the convergence relation of X, X is quasi-normal.

Countable Limits: Since the category of quasi-normal spaces has countable products and subspaces, it has all countable limits.

Countable Coproducts: Let X_i be a sequence of quasi-normal qcb-spaces. The coproduct of X_i formed in QCB has the same convergence relation as the topological coproduct (= direct sum) of the perfectly normal spaces X_i'. Moreover, the latter is perfectly normal by [Engelking 89, Theorem 2.2.7]. So the QCB-coproduct is quasi-normal.

Exponentials: Let X and Y be quasi-normal qcb-spaces. Since X is sequential, the family $\mathcal{C}(\mathsf{X},\mathsf{Y})$ of continuous functions from X to Y is equal to the family $\mathcal{C}(\mathsf{X},\mathsf{Y}')$ of continuous functions from X to the regular space Y'. By Lemma 4.2.2 in [Schröder 03], the compact open topology τ_{co} on $\mathcal{C}(\mathsf{X},\mathsf{Y}')$ induces the convergence relation of continuous convergence. This is the convergence relation of the exponential Y^{X} formed in QCB. So $seq(\tau_{\mathrm{co}})$ is the topology of Y^{X}. By Theorem 3.4.13 in [Engelking 89], τ_{co} is regular. Hence Y^{X} is quasi-normal by Proposition 2.

Exponential Ideal: The above proof shows that Y^{X} is quasi-normal, if Y is quasi-normal and X is a qcb-space. So for every qcb-space X the function space $[0,1]^{([0,1]^{\mathsf{X}})}$ formed in QCB is in QN. Let $e_{\mathsf{X}}\colon \mathsf{X} \to [0,1]^{([0,1]^{\mathsf{X}})}$ be the continuous function given by $e_{\mathsf{X}}(x)(h) := h(x)$. We define $\mathsf{F}(\mathsf{X})$ to be the quasi-normal QCB-subspace of $[0,1]^{([0,1]^{\mathsf{X}})}$ with $e_{\mathsf{X}}(\mathsf{X})$ as its carrier set. It is not difficult to verify that F constitutes a functor that is left adjoint to the inclusion functor of QN into QCB; moreover, Y^{X} is isomorphic to $\mathsf{F}(\mathsf{Y}^{\mathsf{X}})$ in QCB for every space $\mathsf{Y} \in \mathsf{QN}$. So QN is an exponential ideal of QCB.

Countable Colimits: Since the inclusion functor of QN into QCB has a left adjoint (see above) and QCB has all countable colimits (cf. [Schröder 03]), QN has all countable colimits. □

Since any separable metrisable space is a quasi-normal qcb-space, Theorem 6 yields a big supply of quasi-normal qcb-spaces. For example, the aforementioned qcb-spaces $\mathbb{N}^{(\mathbb{N}^{\mathbb{N}})}$ and $\mathbb{R}^{(\mathbb{R}^{\mathbb{R}})}$ are quasi-normal.

Note that topological quotients of quasi-normal spaces need not be quasi-normal: The space N of the natural numbers equipped with the co-final topology is a countably based T_1-space. Thus it is a quotient of some subspace of the Baire space. However, it is not quasi-normal, because it is not even a Hausdorff space. Moreover, its quasi-normal reflection $\mathsf{F}(\mathsf{N})$ has only one point.

Remark. One can prove that QN also inherits its cartesian closed structure from the category of Tychonoff k_{R}-spaces. G. Lukács showed that this category is cartesian closed (cf. [Lukács 04]).

4 An Extension Theorem for Quasi-Normal Qcb-Spaces

In this section we prove an Extension Theorem for quasi-normal qcb-spaces. It states that every continuous function from a functionally closed subset into the unit interval can be extended to a continuous function on the whole space.

4.1 A transitivity property for zero-sets

It is well-known that the subspace operator on topological spaces has the following transitivity property: Any functionally open subset of a functionally open subspace is functionally open in the original space, whereas the analogous statement for functionally closed sets is false in general (cf. [Engelking 89, 2.1.B]).

Validity of the transitivity property for zero-sets (= functionally closed sets) is related to extendability of continuous functions: Consider a functionally closed subspace X of a topological space Y. If any continuous $[0,1]$-valued function on X is extendable onto Y, then any functionally closed subset M of X is functionally closed in Y: Take continuous functions $f\colon \mathsf{X} \to [0,1]$ and $g\colon \mathsf{Y} \to [0,1]$ with

$f^{-1}\{0\} = M$ and $g^{-1}\{0\} = \mathsf{X}$ and extend f to a continuous function $F\colon \mathsf{Y} \to [0,1]$. Then $\lambda_{y \in \mathsf{Y}}.\max\{F(y), g(y)\}$ is a continuous function witnessing that M is functionally closed in Y.

The reverse implication is known to be true as well (see [Engelking 89, 2.1.J]). So we will prove at first that quasi-normal qcb-spaces have the transitivity property for zero-sets:

Proposition 7. *Let* X *be a functionally closed subspace of a quasi-normal qcb-space* Y*. Then every set that is functionally closed in* X *is functionally closed in* Y*. Moreover,* $\mathcal{FO}(\mathsf{X})$ *is the subspace topology induced by* $\mathcal{FO}(\mathsf{Y})$ *on the set* X*.*

4.2 Proof of the transitivity property for zero-sets

Let Y be a quasi-normal qcb-space and X be a functionally closed subspace of Y. By Proposition 4, we can choose a countable pseudobase $\mathcal{B}$ for Y consisting of functionally closed sets.

We define τ to be the topology on Y given by

$$\tau := \big\{ U \in \mathcal{O}(\mathsf{Y}) \,\big|\, U \cap \mathsf{X} \in \mathcal{FO}(\mathsf{X}) \text{ and } U \setminus \mathsf{X} \in \mathcal{FO}(\mathsf{Y}) \big\} \tag{1}$$

and show that τ is equal to the topology $\mathcal{FO}(\mathsf{Y})$ of functionally open sets of Y. Note that τ is indeed closed under arbitrary union, because $\mathcal{O}(\mathsf{Y})$, $\mathcal{FO}(\mathsf{X})$ and $\mathcal{FO}(\mathsf{Y})$ are all hereditarily Lindelöf topologies by having a countable pseudobase.

Clearly we have $\mathcal{FO}(\mathsf{Y}) \subseteq \tau$. The proof of the reverse inclusion $\mathcal{FO}(\mathsf{Y}) \supseteq \tau$ is based on three lemmas about $\mathcal{G}_\delta$-sets, namely on Lemmas 5, 8, and 9. They are direct consequences of the existence of a countable functionally closed pseudobase for Y.

Lemma 8. *Let* V *be open in* Y *and let* $\{\beta_i \,|\, i \in \mathbb{N}\}$ *be a pseudo-open decomposition of* V *into pseudobase elements in* $\mathcal{B}$*. Moreover, let* $(U_j)_j$ *be a sequence of open sets such that* $i \leq j$ *implies* $\beta_i \subseteq U_j$*. Then the* $\mathcal{G}_\delta$*-set* $V \cap \bigcap_{j=0}^{\infty} U_j$ *is open in* Y*.*

Proof. Let $(y_n)_n$ be sequence that converges to some element y_∞ in $V \cap \bigcap_{j=0}^{\infty} U_j$. Then there is some $i \in \mathbb{N}$ such that $(y_n)_n$ is eventually in $\beta_i \subseteq \bigcap_{j=i}^{\infty} U_j$. By being an open neighbourhood of y_∞, the intersection $V \cap \bigcap_{j=0}^{i} U_i$ contains $(y_n)_n$ for almost all n. Thus $(y_n)_n$ is eventually in the set $V \cap \bigcap_{j=0}^{\infty} U_j$. We conclude that $V \cap \bigcap_{j=0}^{\infty} U_j$ is sequentially open and thus open, because Y is sequential. □

The complement of any closed subset of Y has a decompostion into sets of the countable and functionally closed pseudobase $\mathcal{B}$. Hence:

Lemma 9. *Every closed subset of* Y *is a functional* $\mathcal{G}_\delta$*-set of* Y*.*

The key step of the proof of the transitivity property for zero-sets (Proposition 7) is to show that the topology τ satisfies the following weak normality property.

Lemma 10. *For every functionally closed set* $A \in \mathcal{FA}(\mathsf{Y})$ *and every set* $U \in \tau$ *containing* A *there is a set* $U' \in \tau$ *and a functionally closed set* $A' \in \mathcal{FA}(\mathsf{Y})$ *satisfying* $A \subseteq U' \subseteq A' \subseteq U$.

Proof. Since $A \cap \mathsf{X}$ and $\mathsf{X} \setminus (U \cap \mathsf{X})$ are disjoint and functionally closed in X, there are a functionally open set $W \in \mathcal{FO}(\mathsf{X})$ and a functionally closed set $F \in \mathcal{FA}(\mathsf{X})$ such that $A \cap \mathsf{X} \subseteq W \subseteq F \subseteq U \cap \mathsf{X}$.

Let $V := W \cup (U \setminus \mathsf{X})$ and $G := F \cup (U \setminus \mathsf{X})$. Then $A \subseteq V \subseteq G \subseteq U$. Since $W \cup (\mathsf{Y} \setminus \mathsf{X})$ is open in Y and $V = (W \cup (\mathsf{Y} \setminus \mathsf{X})) \cap U$, V is open in Y. Thus there exists a pseudo-open decomposition $\{\beta_i \mid i \in \mathbb{N}\}$ of V into sets of the functionally closed pseudobase $\mathcal{B}$. By Lemma 9 the closed set F and hence G are functional $\mathcal{G}_\delta$-sets in Y. Therefore there are functionally open sets $G_0, G_1, \ldots \in \mathcal{FO}(\mathsf{Y})$ with $G = \bigcap_{j=0}^{\infty} G_j$.

Since $A \cup \bigcup_{i=0}^{n} \beta_i$ and $\mathsf{Y} \setminus G_n$ are disjoint functionally closed sets, there is a continuous function $h_n \colon \mathsf{Y} \to [0,1]$ such that $h_n^{-1}\{0\} = A \cup \bigcup_{i=0}^{n} \beta_i$ and $h_n^{-1}\{1\} = \mathsf{Y} \setminus G_n$ for every $n \in \mathbb{N}$. The functionally open set $O_n := h_n^{-1}[0, 1/2)$ and the functionally closed set $A_n := h_n^{-1}[0, 1/2]$ satisfy $A \cup \bigcup_{i=0}^{n} \beta_i \subseteq O_n \subseteq A_n \subseteq G_n$. By Lemma 8, the set $U' := V \cap \bigcap_{n=0}^{\infty} O_n$ is open. Clearly the set $A' := \bigcap_{n=0}^{\infty} A_n$ is functionally closed in Y. The pair (U', A') satisfies

$$A \subseteq V \cap \bigcap_{n=0}^{\infty} O_n = U' \subseteq A' = \bigcap_{n=0}^{\infty} A_n \subseteq \bigcap_{n=0}^{\infty} G_n = G = F \cup (U \setminus \mathsf{X}) \subseteq U\,.$$

As $U' \cap \mathsf{X} = W \cap \bigcap_{j=0}^{\infty} (O_j \cap \mathsf{X})$ is both an open set and a functional $\mathcal{G}_\delta$-set of X, the set $U' \cap \mathsf{X}$ is functionally open in X by Lemma 5. Similarly, as the set $U' \setminus \mathsf{X}$ is open in Y and equal to the functional $\mathcal{G}_\delta$-set $(U \setminus \mathsf{X}) \cap \bigcap_{j=0}^{\infty} O_j \in \mathcal{FG}(\mathsf{Y})$, it is functionally open in Y by Lemma 5. Hence $U' \in \tau$. □

We employ Lemma 10 to show the following separation lemma. It resembles Urysohn's Separation Lemma which states that two disjoint closed sets in a normal space can be separated by a continuous real-valued function (see e.g. Theorem 1.5.11 in [Engelking 89]).

Lemma 11. *For every functionally closed set* $A \in \mathcal{FA}(\mathsf{Y})$ *and every set* $U \in \tau$ *with* $A \subseteq U$ *there is a continuous function* $h \colon \mathsf{Y} \to [0,1]$ *with* $A \subseteq h^{-1}\{0\}$ *and* $\mathsf{Y} \setminus U \subseteq h^{-1}\{1\}$.

Proof. We use the construction idea of a standard proof of Urysohn's Separation Lemma. Let $D_0 := \{0, 1\}$ and $D_n := \{(2i-1)/2^n \mid i \in \{1, \ldots, 2^{n-1}\}\}$ for $n \geq 1$. For each dyadic rational d in $D := \bigcup_{n=0}^{\infty} D_n$ we define inductively an open set $U_d \in \tau$ and a closed set $A_d \in \mathcal{FA}(\mathsf{Y})$ such that

$$c < e \text{ implies } U_c \subseteq A_c \subseteq U_e \subseteq A_e\,. \tag{2}$$

"$n = 0$": We set $U_0 := \emptyset$, $A_0 := A$, $U_1 := U$ and $A_1 := \mathsf{Y}$.

"$n > 0$": For $d \in D_n$ we have $\{d - 2^{-n}, d + 2^{-n}\} \subseteq \bigcup_{i=0}^{n-1} D_i$. By the induction hypothesis we have already defined a pair $(A_{d-2^{-n}}, U_{d+2^{-n}})$ satisfying $A_{d-2^{-n}} \subseteq U_{d+2^{-n}}$. We apply Lemma 10 to this pair to obtain sets $U' \in \tau$ and $A' \in \mathcal{A}(\mathsf{Y})$ satisfying $A_{d-2^{-n}} \subseteq U' \subseteq A' \subseteq U_{d+2^{-n}}$. We set $U_d := U'$ and $A_d := A'$.

Clearly, the sequence $(U_d, A_d)_{d \in D}$ satisfies (2). We define $h, h' \colon \mathsf{Y} \to [0,1]$ by

$$h(y) := \inf\{d \in D \,|\, y \in A_d\} \ \text{ and } \ h'(y) := \sup\{e \in D \,|\, y \in \mathsf{Y} \setminus U_e\}$$

and show $h(y) = h'(y)$ for all $y \in \mathsf{Y}$. If $h(y) > h'(y)$, then there would be some $d \in D$ with $h(y) > d > h'(y)$ implying $y \notin A_d \wedge y \in U_d$ and contradicting $U_d \subseteq A_d$. On the other hand, if $h(y) < h'(y)$, then there would be some $d \in D$ such that $h(y) < d < h'(y) \wedge y \in A_d$ and some $e \in D$ such that $d < e < h'(y) \wedge y \in \mathsf{Y} \setminus U_e$, contradicting $A_d \subseteq U_e$. For $y \in \mathsf{Y}$ and $s, t \in D$ we deduce from $h = h'$ the two implications

$$s < h(y) < t \implies y \in U_t \setminus A_s \implies s \leq h(y) \leq t\,.$$

From both implications it follows that h is continuous. □

From Lemma 11 we can deduce:

Lemma 12. *The topology τ is equal to the family of functionally open sets of Y.*

Proof. We have already observed the inclusion $\mathcal{FO}(\mathsf{Y}) \subseteq \tau$. Now let $U \in \tau$. By being open, U has a pseudo-open decomposition $\{\beta_i \,|\, i \in \mathbb{N}\}$ consisting of functionally closed pseudobase elements. By Lemma 11, for every i there is a topologically continuous function $h_i \colon \mathsf{Y} \to [0,1]$ with $\beta_i \subseteq h_i^{-1}\{0\}$ and $\mathsf{Y} \setminus U \subseteq h_i^{-1}\{1\}$. By [Engelking 89, Theorem 1.4.7], the function $f \colon \mathsf{Y} \to [0,1]$ defined by $f(y) := 1 - \sum_{i=0}^{\infty} 2^{-i-1} \cdot h_i(y)$ is continuous. Clearly, $f^{-1}\{0\} = \mathsf{Y} \setminus U$. □

Lemma 12 finally implies the transitivity property for functionally closed sets in quasi-normal spaces stated in Proposition 7.

Proof. (Proposition 7) Let $A \in \mathcal{FA}(\mathsf{X})$. Then $U := \mathsf{Y} \setminus A = (\mathsf{X} \setminus A) \cup (\mathsf{Y} \setminus \mathsf{X})$ is an element of τ and thus functionally open in Y by Lemma 12. Hence $A \in \mathcal{FA}(\mathsf{Y})$. Conversely, for every $B \in \mathcal{FA}(\mathsf{Y})$ the set $B \cap \mathsf{X}$ is functionally closed in X. Thus $\mathcal{FO}(\mathsf{X})$ is the subspace topology induced by $\mathcal{FO}(\mathsf{Y})$ on the subset X. □

4.3 The Extension Theorem for Continuous Functions

In this section we formulate and prove the Extension Theorem for quasi-normal qcb-spaces. The proof follows the lines of the proof of the original Tietze-Urysohn Theorem (cf. [Engelking 89, Theorem 2.1.8]), using Proposition 7 in place of

Urysohn's Separation Lemma. Although Theorem 13 can be deduced from the original Tietze-Urysohn Theorem with the help of Proposition 7, we give an explicit construction, because the latter can be easily enhanced to a proof of the Extension Theorem for computable functions in Section 5.

Theorem 13. *Let* X *be a functionally closed subspace of a quasi-normal qcb-space* Y*.*

1. *Every continuous function* $f\colon \mathsf{X} \to [0,1]$ *can be extended to a continuous function* $F\colon \mathsf{Y} \to [0,1]$ *satisfying* $F(x) = f(x)$ *for all* $x \in \mathsf{X}$*.*
2. *Every continuous function* $f\colon \mathsf{X} \to \mathbb{R}$ *can be extended to a continuous function* $F\colon \mathsf{Y} \to \mathbb{R}$ *satisfying* $F(x) = f(x)$ *for all* $x \in \mathsf{X}$*.*

Proof. 1. We modify the proof of the Tietze-Urysohn Theorem in [Engelking 89]. For the sake of simplicity, we show the statement for functions into the interval $[-1,1]$ in place of $[0,1]$. First we prove that the multi-function Φ which maps a dyadic number c in $(0,1]$ and a continuous function $h\colon \mathsf{X} \to [-1,1]$ with $\sup_{x\in\mathsf{X}} |h(x)| \leq c$ to all continuous functions $H\colon \mathsf{Y} \to [-1,1]$ satisfying

$$\sup_{y\in\mathsf{Y}} \big|H(y)\big| \leq \tfrac{5}{16}c \quad \text{and} \quad \sup_{x\in\mathsf{X}} \big|h(x) - H(x)\big| \leq \tfrac{11}{16}c \tag{3}$$

is total. Since $A := h^{-1}[-c, -\frac{3}{8}c]$ and $B := h^{-1}[\frac{3}{8}c, c]$ are functionally closed in X, they are functionally closed in Y by Proposition 7. So there exists a continuous function $k\colon \mathsf{Y} \to [0,1]$ satisfying $k^{-1}\{0\} = A$ and $k^{-1}\{1\} = B$, because A and B are disjoint. We define the continuous function $H\colon \mathsf{Y} \to [-1,1]$ by $H(y) := \frac{5}{8}c\big(k(y) - \frac{1}{2}\big)$. One easily verifies that H satisfies Condition (3), hence $H \in \Phi(c,h)$.

Now we construct inductively a sequence $(g_i)_i$ of continuous functions $g_i\colon \mathsf{Y} \to [-1,1]$ such that

$$\sup_{y\in\mathsf{Y}} \big|g_i(y)\big| \leq \tfrac{5}{16} \cdot (\tfrac{11}{16})^i \quad \text{and} \quad \sup_{x\in\mathsf{X}} \Big|f(x) - \sum_{j=0}^{i} g_j(x)\Big| \leq (\tfrac{11}{16})^{i+1} \tag{4}$$

by choosing $g_0 \in \Phi(1,f)$ and $g_{i+1} \in \Phi\big((\frac{11}{16})^{i+1},\, \lambda_{x\in\mathsf{X}}.\, f(x) - \sum_{j=0}^{i} g_i(x)\big)$. Note that the second statement in (4) ensures that the multi-function Φ is applied to a legal argument. For all $n \in \mathbb{N}$ we have

$$\Big|\sum_{j\geq 2n} g_j(y)\Big| \leq \tfrac{5}{16} \cdot (\tfrac{11}{16})^{2n} \cdot \sum_{j=0}^{\infty} (\tfrac{11}{16})^j = (\tfrac{121}{256})^n \cdot \tfrac{5}{16} \cdot \tfrac{1}{1-11/16} \leq 2^{-n}\,.$$

Therefore the formula $F(y) := \sum_{j=0}^{\infty} g_j(y)$ defines a function from Y into the interval $[-1,1]$. Moreover, the function sequence $\big(\lambda_{y\in\mathsf{Y}}.\sum_{j=0}^{2n} g_j(y)\big)_n$ converges uniformly to F. Hence F is a continuous function. The second statement in (4) implies $F(x) = f(x)$ for all $x \in \mathsf{X}$.

2. Let X' and Y' be the topological spaces that carry the topologies $\mathcal{FO}(\mathsf{X})$ and $\mathcal{FO}(\mathsf{Y})$, respectively. By Proposition 7, X' is a topological subspace of Y'. As Y' is normal and f is topologically continuous w.r.t. $\mathcal{FO}(\mathsf{X})$, the statement follows from the original Tietze-Urysohn Theorem. □

5 An Effective Version of the Extension Theorem

In this section we establish an effective version of the Tietze-Urysohn Extension Theorem. This theorem is formulated for qcb-spaces that satisfy a computable notion of quasi-normality, which we call *effective quasi-normality.*

5.1 Representations for families of subsets

Given an admissible representation δ of a qcb-space Y, we introduce at first representations (derived from δ) for the following families of subsets of Y: the open sets, the closed sets, the functionally opens sets, the functionally closed sets, and the functional G_δ-sets (= countable intersections of functionally open sets).

To define the representations of $\mathcal{O}(\mathsf{Y})$ and $\mathcal{A}(\mathsf{Y})$, we use the fact that every open set has a pseudo-open decomposition into elements of the pseudobase $\mathcal{B}_\delta$ induced by δ. Using the effective bijective numbering $\mathsf{w}\colon \mathbb{N} \to \mathbb{N}^*$ of $\mathbb{N}^*$ from Section 2.1, we define the representations $\delta^{\mathcal{O}}$ of $\mathcal{O}(\mathsf{Y})$ and $\delta^{\mathcal{A}}$ of $\mathcal{A}(\mathsf{Y})$ by

$$\delta^{\mathcal{O}}(q) = V \;:\Longleftrightarrow\; \delta^{-1}(V) = \big\{p \in \mathrm{dom}(\delta) \,\big|\, \exists i.\, q(i) > 0 \wedge \mathsf{w}(q(i)-1) \sqsubseteq p\big\}$$

and $\delta^{\mathcal{A}}(q) := \mathsf{Y} \setminus \delta^{\mathcal{O}}(q)$. So for $q \in \mathrm{dom}(\delta^{\mathcal{O}})$ we have $\delta^{\mathcal{O}}(q) = \bigcup_{i\in\mathbb{N}} B_\delta(q(i))$. Moreover, $\{B_\delta(q(i)) \,|\, i \in \mathbb{N}\}$ is a pseudo-open decomposition of $\delta^{\mathcal{O}}(q)$: admissibility of δ implies that for all sequences $(y_n)_n$ converging to some element $y_\infty \in \delta^{\mathcal{O}}(q)$ there are δ-names p_n for y_n such that $\lim\limits_{n\to\infty} p_n = p_\infty$ (cf. [Schröder 02]). There is some $i \in \mathbb{N}$ such that $\mathsf{w}(q(i)-1)$ is a prefix of p_∞ and thus a prefix of almost all p_n, implying that $(y_n)_n$ is eventually in $B_\delta(q(i))$.

One can prove that $\delta^{\mathcal{O}}$ is computably equivalent to the Sierpiński representation of $\mathcal{O}(\mathsf{Y})$, which encodes an open set V via its characteristic function cf_V from Y into the Sierpiński space defined by $cf_V(y) = \top :\Longleftrightarrow y \in V$. The Sierpiński space has $\{\bot, \top\}$ as its underlying set and $\{\bot\}$ as its only closed singleton.

By using the standard function representation, $[\delta \to \varrho_{[0,1]}]$, of the set of continuous functions from Y to $[0,1]$, we define representations $\delta^{\mathcal{FO}}$ of $\mathcal{FO}(\mathsf{Y})$ and $\delta^{\mathcal{FA}}$ of $\mathcal{FA}(\mathsf{Y})$ straightforwardly by

$$\delta^{\mathcal{FA}}(q) := \big\{y \in \mathsf{Y} \,\big|\, [\delta \to \varrho_{[0,1]}](q)(y) = 0\big\} \text{ and } \delta^{\mathcal{FO}}(q) := \mathsf{Y} \setminus \delta^{\mathcal{FA}}(q)$$

for $q \in \mathrm{dom}([\delta \to \varrho_{[0,1]}])$. Finally, we define the representation $\delta^{\mathcal{FG}}$ of the family of functional $\mathcal{G}_\delta$-sets by

$$\delta^{\mathcal{FG}}(\langle q_0, q_1, \dots \rangle) := \bigcap_{j=0}^{\infty} \delta^{\mathcal{FO}}(q_j)\,,$$

where $\langle\,.\,\rangle$ denotes a standard computable bijection from $(\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}$ to $\mathbb{N}^{\mathbb{N}}$.

With standard methods of TTE, one can prove the following lemma. It formulates effective versions of known theorems in the theory of topological spaces.

Lemma 14.

1. *Finite union and finite intersection (on the respective family of subsets) are computable w.r.t. each of the representations* $\delta^{\mathcal{O}}, \delta^{\mathcal{A}}, \delta^{\mathcal{FO}}, \delta^{\mathcal{FA}}, \delta^{\mathcal{FG}}$.

2. *Countable intersection of closed subsets is computable w.r.t.* $[\varrho_{\mathbb{N}} \to \delta^{\mathcal{A}}]$ *and* $\delta^{\mathcal{A}}$*; countable intersection of functionally closed subsets is computable w.r.t.* $[\varrho_{\mathbb{N}} \to \delta^{\mathcal{FA}}]$ *and* $\delta^{\mathcal{FA}}$.

3. *The multi-function that maps two disjoint functionally closed sets* A, B *to all continuous functions* $h\colon \mathsf{Y} \to [0,1]$ *satisfying* $h^{-1}\{0\} = A$ *and* $h^{-1}\{1\} = B$ *is computable w.r.t.* $\delta^{\mathcal{FA}}$ *and* $[\delta \to \varrho_{[0,1]}]$.

4. *The function that maps a continuous function* $h\colon \mathsf{Y} \to \mathbb{R}$ *and two real numbers* $r, s \in \mathbb{R}$ *to the functionally closed set* $h^{-1}[r,s]$ *is computable w.r.t.* $[\delta \to \varrho_{\mathbb{R}}]$, $\varrho_{\mathbb{R}}$ *and* $\delta^{\mathcal{FA}}$.

5. *The representation* $\delta^{\mathcal{FO}}$ *is computably reducible to* $\delta^{\mathcal{O}}$*, and* $\delta^{\mathcal{FA}}$ *is computably reducible to* $\delta^{\mathcal{A}}$.

6. *For any* $\delta^{\mathcal{A}}$*-computable closed subset* M*, the function* $O \mapsto O \cup \mathsf{Y} \setminus M$ *is computable w.r.t.* $(\delta|^{M})^{\mathcal{O}}$ *and* $\delta^{\mathcal{O}}$.

7. *Let* $\cdot^{\mathsf{op}}$ *be any operator in* $\{\cdot^{\mathcal{O}}, \cdot^{\mathcal{A}}, \cdot^{\mathcal{FO}}, \cdot^{\mathcal{FA}}\}$*. For any* δ^{op}*-computable subset* M*, the function* $A \mapsto A \cap M$ *is computable w.r.t.* δ^{op} *and* $(\delta|^{M})^{\mathsf{op}}$.

8. *Let* $\cdot^{\mathsf{op}}$ *be any operator in* $\{\cdot^{\mathcal{O}}, \cdot^{\mathcal{A}}, \cdot^{\mathcal{FO}}\}$*. For any* δ^{op}*-computable subset* M*,* $(\delta|^{M})^{\mathsf{op}}$ *is computably reducible to* δ^{op}.

Here $\delta|^{M}$ denotes the corestriction of δ to the subset M. Remember that a representation ϕ_1 for a set M is computably reducible to a representation ϕ_2 for M or a superset of M, if there is computable function g satisfying $\phi_2(g(p)) = \phi_1(p)$ for all $p \in \mathrm{dom}(\phi_1)$. Two representations are computably equivalent, if there are computable reducible to each other (cf. [Weihrauch 00]).

5.2 Effectively quasi-normal spaces

We introduce an effectivised version of the notion of a quasi-normal qcb-space. The idea is to use the fact that quasi-normal qcb-spaces have an admissible representation δ such that the pseudobase $\mathcal{B}_\delta = \{B_\delta(n) \mid n \in \mathbb{N}\}$ induced by δ (see Section 2.5) is functionally closed. This follows from Proposition 4 and the fact that the standard construction of an admissible representation (see [Schröder 02]) from a pseudobase closed under finite intersection induces this pseudobase.

Definition 15. Let Y be a qcb-space.

1. An admissible representation δ of Y is called *effectively functionally closed*, if the pseudobase $\mathcal{B}_\delta = \{B_\delta(n) \mid n \in \mathbb{N}\}$ induced by δ consists of functionally closed sets and the sequence $(B_\delta(n))_n$ is computable w.r.t. the representation $\delta^{\mathcal{FA}}$.
2. The space Y is called *effectively quasi-normal*, if Y has an admissible and effectively functionally closed representation.

By having a functionally closed pseudobase, an effectively quasi-normal space is indeed quasi-normal (cf. Proposition 4). An example of an effectively functionally closed representation is the signed-digit representation $\varrho_\mathbb{R}$, because the function $(a,b) \mapsto [a,b]$ is computable w.r.t. $\varrho_\mathbb{R}$ and $\varrho_\mathbb{R}^{\mathcal{FA}}$. Computable equivalence of representations do not preserve this effectivity property, simply because there are effective representations of the Euclidean space that induce pseudobases containing non-closed sets.

5.3 The effective Tietze-Urysohn Extension Theorem

Now we are ready to formulate the effective Tietze-Urysohn Extension Theorem for effectively quasi-normal qcb-spaces. We state a non-uniform and a uniform version.

Theorem 16. *Let Y be a quasi-normal qcb-space equipped with an admissible effectively functionally closed representation δ. Moreover, let X be a $\delta^{\mathcal{FA}}$-computable subset of Y. Then every $(\delta|^X, \varrho_{[0,1]})$–computable function $f\colon X \to [0,1]$ has a $(\delta, \varrho_{[0,1]})$–computable extension $F\colon \mathsf{Y} \to [0,1]$.*

Theorem 16 follows from the uniform version:

Theorem 17. *Let Y be a quasi-normal qcb-space equipped with an admissible effectively functionally closed representation δ. Moreover, let X be a $\delta^{\mathcal{FA}}$-computable subset of Y. Then the multi-function that maps any continuous function $f\colon X \to [0,1]$ to all its continuous extensions $F\colon \mathsf{Y} \to [0,1]$ is computable w.r.t. $[\delta|^X \to \varrho_{[0,1]}]$ and $[\delta \to \varrho_{[0,1]}]$.*

One can even show that extendability of continuous functions is also computable uniformly in the functionally closed domain. We omit the details.

5.4 Sketch of Proof of the effective Extension Theorem

The effective Tietze-Urysohn Theorem 17 can be deduced from the following proposition along with Lemma 14 by carefully effectivising the proof of Theorem 13.

Proposition 18. *Let* Y *be a quasi-normal qcb-space equipped with an admissible effectively functionally closed representation* δ*. For every* $\delta^{\mathcal{FA}}$*-computable subset* $X \subseteq \mathsf{Y}$*, the representation* $(\delta|^X)^{\mathcal{FA}}$ *is computably reducible to* $\delta^{\mathcal{FA}}$*.*

We prove Proposition 18 by showing effective versions of the lemmas in Section 4 on which Proposition 7 is based. Let δ be an admissible effectively functionally closed representation of Y and let X be a $\delta^{\mathcal{FA}}$-computable subset of Y. Lemma 14.7 implies that $\delta|^X$ is an effectively functionally closed representation of the space X endowed with the (sequential) subspace topology inherited from Y. As a pseudobase for Y we use the functionally closed pseudobase $\mathcal{B}_\delta$ induced by δ.

At first we introduce a representation Ω of the topology τ from Equation (1). We define Ω by

$$\Omega(\langle q,r,s\rangle) = U \;:\Longleftrightarrow\; \big(\delta^{\mathcal{O}}(q) = U,\, (\delta|^X)^{\mathcal{FO}}(r) = U \cap X,\, \delta^{\mathcal{FO}}(s) = U \setminus X\big)\,.$$

Here $\langle \cdot,\cdot,\cdot\rangle$ denotes a computable bijection between $\mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ and $\mathbb{N}^{\mathbb{N}}$.

Our effective version of Lemma 5 states that any $\delta^{\mathcal{O}}$-name of a functionally open set V can be converted into a $\delta^{\mathcal{FO}}$-name, when additionally the information about V as a functional $\mathcal{G}_\delta$-set is given by means of a $\delta^{\mathcal{FG}}$-name. To formulate this statement precisely, we represent the family of all open functional $\mathcal{G}_\delta$-sets by the conjunction of the representations $\delta^{\mathcal{O}}$ and $\delta^{\mathcal{FG}}$. The conjunction $\delta^{\mathcal{O}} \wedge \delta^{\mathcal{FG}}$ is defined by $(\delta^{\mathcal{O}} \wedge \delta^{\mathcal{FG}})(\langle q,s\rangle) = V :\Longleftrightarrow V = \delta^{\mathcal{O}}(q) = \delta^{\mathcal{FG}}(s)$, cf. [Schröder 03, Weihrauch 00].

Lemma 19. *The representation* $\delta^{\mathcal{O}} \wedge \delta^{\mathcal{FG}}$ *is computably reducible to* $\delta^{\mathcal{FO}}$*, and* $(\delta|^X)^{\mathcal{O}} \wedge (\delta|^X)^{\mathcal{FG}}$ *is computably reducible to* $(\delta|^X)^{\mathcal{FO}}$*.*

Proof. We only consider the slightly more difficult case $\phi := \delta|^X$. Let $\langle q,s\rangle \in \mathrm{dom}(\phi^{\mathcal{O}} \wedge \phi^{\mathcal{FG}})$ and let $V := \phi^{\mathcal{O}}(q) = \phi^{\mathcal{FG}}(s)$. We can compute $s_0, s_1, \ldots \in \mathrm{dom}(\phi^{\mathcal{FO}})$ with $\langle s_0, s_1, \ldots\rangle = s$. Since δ is an effectively functionally closed representation, the map $i \mapsto B_\phi(i) = B_\delta(i) \cap X$ is computable w.r.t. $\phi^{\mathcal{FA}}$. Thus for every $n \in \mathbb{N}$ we can compute a $\phi^{\mathcal{FA}}$-name of the closed set $F^q_n = \bigcup_{i=0}^n B_\phi(q(i)) \subseteq$

V by Lemma 14. Since F^q_n and $\phi^{\mathcal{FA}}(s_n)$ are disjoint, by Lemma 14 we can construct for each $n \in \mathbb{N}$ a function $h_{q,s,n}\colon X \to [0,1]$ satisfying $h^{-1}_{q,s,n}\{0\} = \phi^{\mathcal{FA}}(s_n)$ and $h^{-1}_{q,s,n}\{1\} = F^q_n$.

We define a function $f_{q,s}\colon X \to [0,1]$ by $f_{q,s}(x) := \inf_{n\in\mathbb{N}} h_{q,s,n}(x)$ and show that $(\hat{q}, \hat{s}) \mapsto f_{\hat{q},\hat{s}}$ is computable w.r.t. $[\phi \to \varrho_{[0,1]}]$. Let $p \in \mathrm{dom}(\phi)$ and $k \in \mathbb{N}$. Set $x := \phi(p)$. In order to compute a dyadic number approximating $f_{q,s}(x)$ with precision 2^{-k}, we apply exhaustive search to find a pair $(i,m) \in \mathbb{N}^2$ such that either $\mathsf{w}(q(i)-1)$ is a prefix of p or the prefices $q^{<m}, s^{<m}, p^{<m}$ admit the verification of $h_{q,s,i}(\phi(p)) \leq 2^{-k}$. In the former case, we have $x \in B_\phi(q(i)) \subseteq V$ and thus $h_{q,s,j}(x) = 1$ for all $j \geq i$, so that it suffices to compute a dyadic approximation to $\min\{h_{q,s,0}(x), \ldots, h_{q,s,i}(x)\}$ with precision 2^{-k}. In the latter case we have $0 \leq f_{q,s}(x) \leq h_{q,s,i}(x) \leq 2^{-k}$, so that 0 is an appropriate approximation. Note that if x belongs to V then some prefix of p has to be listed by the name q. If x is not in V, then there is some j with $x \notin \phi^{\mathcal{FO}}(s_j)$ and hence $h_{q,s,j}(x) = 0$. Therefore the exhaustive search will be eventually successful for $\langle q,s\rangle \in \mathrm{dom}(\phi^{\mathcal{O}} \wedge \phi^{\mathcal{FG}})$ and $p \in \mathrm{dom}(\phi)$.

We conclude that there is computable function $g\colon \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ realising $\langle \hat{q}, \hat{s}\rangle \mapsto f_{\hat{q},\hat{s}}$. For every element $x \in V$ there is some i such that $x \in B_\phi(q(i))$, implying $f_{q,s}(x) = \min\{h_{q,s,0}(x), \ldots, h_{q,s,i}(x)\}$ and thus $f_{q,s}(x) > 0$. For every element $x \notin V$ there is some j with $x \notin \delta^{\mathcal{FO}}(s_j)$ implying $f_{q,s}(x) = h_{q,s,j}(x) = 0$. Hence g translates $\phi^{\mathcal{O}} \wedge \phi^{\mathcal{FG}}$ into $\phi^{\mathcal{FO}}$. □

We effectivise Lemma 8 by introducing a technical representation $\delta^{\mathcal{OB}}$ of $\mathcal{O}(\mathsf{Y})$ and by stating that it is computably reducible to $\delta^{\mathcal{O}}$. We define $\delta^{\mathcal{OB}}$ by

$$\delta^{\mathcal{OB}}(\langle q, s_0, s_1, \ldots\rangle) = V \;:\Longleftrightarrow\; \begin{cases} V = \delta^{\mathcal{O}}(q) \cap \bigcap_{j=0}^{\infty} \delta^{\mathcal{O}}(s_j) \quad \text{and} \\ \forall i \leq j.\, B_\delta(q(i)) \subseteq \delta^{\mathcal{O}}(s_j) \end{cases}$$

for all $q, s_0, s_1, \ldots \in \mathbb{N}^\mathbb{N}$ and $V \in \mathcal{O}(\mathsf{Y})$. Note that $\delta^{\mathcal{O}}(q) = \delta^{\mathcal{OB}}(\langle q,q,q,\ldots\rangle)$.

Lemma 20. *The representation $\delta^{\mathcal{OB}}$ is computably reducible to $\delta^{\mathcal{O}}$.*

Proof. Using a computable bijection $\langle\cdot,\cdot,\cdot\rangle\colon \mathbb{N}^3 \to \mathbb{N}$, we define $g\colon \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ by

$$g(\langle q, s_0, s_1, s_2, \ldots\rangle)(\langle a,b,c\rangle) := \begin{cases} a+1 & \text{if } q(b) > 0 \text{ and } \mathsf{w}(q(b)-1) \text{ is a prefix of } \mathsf{w}(a) \text{ and} \\ & \text{for all } j < q(b) \text{ there is some } k_j \in \{s_j(l) \mid l \leq c\} \\ & \text{such that } k_j > 0 \text{ and } \mathsf{w}(k_j - 1) \text{ is a prefix of } \mathsf{w}(a) \\ 0 & \text{otherwise} \end{cases}$$

for all $q, s_0, s_1, s_2, \ldots \in \mathbb{N}^\mathbb{N}$ and all $a,b,c \in \mathbb{N}$. Clearly g is computable. For all

$\langle q, s_0, s_1, s_2, \dots \rangle \in \mathrm{dom}(\delta^{\mathcal{OB}})$ and $p \in \mathrm{dom}(\delta)$, we have

$$\begin{aligned}
&\delta(p) \in \delta^{\mathcal{OB}}(q)\\
&\iff \exists b \in \mathbb{N}.\, \mathsf{w}(q(b)-1) \sqsubseteq p \,\wedge\, \forall j \in \mathbb{N}. \exists k_j \in s_j(\mathbb{N}).\, \mathsf{w}(k_j - 1) \sqsubseteq p\\
&\iff \exists b \in \mathbb{N}.\big(\mathsf{w}(q(b)-1) \sqsubseteq p \,\wedge\, \forall j < q(b). \exists k_j \in s_j(\mathbb{N}).\, \mathsf{w}(k_j - 1) \sqsubseteq p\big)\\
&\iff \exists a, b, c \in \mathbb{N}.\big(\mathsf{w}(a) \sqsubseteq p \,\wedge\, \mathsf{w}(q(b)-1) \sqsubseteq \mathsf{w}(a)\\
&\qquad\qquad \wedge\, \forall j < q(b). \exists k_j \in \{s_j(l) \,|\, l \leq c\}.\, \mathsf{w}(k_j - 1) \sqsubseteq \mathsf{w}(a)\big)\\
&\iff \exists i \in \mathbb{N}.\, \mathsf{w}\big(g\langle q, s_0, s_1, s_2, \dots \rangle(i) - 1\big) \sqsubseteq p\,.
\end{aligned}$$

Therefore g translates $\delta^{\mathcal{OB}}$ into $\delta^{\mathcal{O}}$. □

The effectivity condition on δ ensures the following effectivisation of Lemma 9.

Lemma 21. *The representation $\delta^{\mathcal{A}}$ is computably reducible to $\delta^{\mathcal{FG}}$.*

Proof. Any $\delta^{\mathcal{A}}$-name q of a closed set A provides a sequence $(\beta_i)_i$ of pseudobase elements in $\mathcal{B}_\delta$ such that their union is the complement of A. By the effectivity condition on δ and by Lemma 14, we can convert q into a $\delta^{\mathcal{FG}}$-name of the set $A = \bigcap_{i=0}^{\infty}(\mathsf{Y} \setminus \beta_i)$. □

Lemmas 10 and 11 can be effectivised by stating computability of appropriate multi-functions.

Lemma 22. *The multi-function which maps a functionally closed set $A \in \mathcal{FA}(\mathsf{Y})$ and a set $U \in \tau$ with $A \subseteq U$ to all pairs $(U', A') \in \tau \times \mathcal{FA}(\mathsf{Y})$ satisfying $A \subseteq U' \subseteq A' \subseteq U$ is computable w.r.t. the representations $\delta^{\mathcal{FA}}$ and Ω.*

Lemma 23. *The multi-function which maps a functionally closed set $A \in \mathcal{FA}(\mathsf{Y})$ and a set $U \in \tau$ with $A \subseteq U$ to all continuous functions $h\colon \mathsf{Y} \to [0,1]$ satisfying $A \subseteq h^{-1}\{0\}$ and $\mathsf{Y} \setminus U \subseteq h^{-1}\{1\}$ is computable w.r.t. the representations $\delta^{\mathcal{FA}}$, Ω and $[\delta \to \varrho_{[0,1]}]$.*

By Lemma 12, the topology is τ is equal to $\mathcal{FO}(\mathsf{Y})$. We express this property in terms of computable equivalence of representations.

Lemma 24. *The representations Ω and $\delta^{\mathcal{FO}}$ are computably equivalent.*

Lemmas 22, 23 and 24 can be proven by effectivisations of the proofs of their topological counterparts using Lemmas 14, 19, 20 and 21. As a pseudo-open decomposition into functionally closed sets for an open subset V of Y given by an $\delta^{\mathcal{O}}$-name q, one uses the family $\{\mathcal{B}_\delta(q(i)) \,|\, i \in \mathbb{N}\}$. We omit the details.

We now show how to effectivise the proof of Proposition 7 using Lemma 14 and the representations of the relevant families of subsets of Y.

Proof. (Proposition 18) Let $t_X \in \mathbb{N}^{\mathbb{N}}$ be computable such that $\delta^{\mathcal{FA}}(t_X) = X$. Lemma 14.5 and 14.6 imply that there is a computable function $g_1 :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ realising the function $h\colon \mathcal{FA}(X) \to \mathcal{O}(\mathsf{Y})$ defined by $h(A) := (X \setminus A) \cup (\mathsf{Y} \setminus X)$ w.r.t. $(\delta|^X)^{\mathcal{FA}}$ and $\delta^{\mathcal{O}}$. By Lemma 24, there is a computable function g_2 translating Ω into $\delta^{\mathcal{FO}}$. The computable function $g :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ defined by $g(r) := g_2(\langle g_1(r), r, t_X\rangle)$ translates $(\delta|^X)^{\mathcal{FA}}$ into $\delta^{\mathcal{FA}}$, because $\langle g_1(r), r, t_X\rangle$ is an Ω-name of the open complement $h\big((\delta|^X)^{\mathcal{FA}}(r)\big)$ of the functionally closed set $(\delta|^X)^{\mathcal{FA}}(r)$ for every $r \in \mathrm{dom}((\delta|^X)^{\mathcal{FA}})$.

6 Discussion

We have shown that quasi-normality yields a reasonable substitute for the property of normality in the category of qcb-spaces. It admits a continuous and, in its effective version, a computable Extension Theorem for functions defined on functionally closed subspaces. The category QN of quasi-normal qcb-spaces contains all countably based normal spaces and enjoys excellent closure properties, e.g. QN is cartesian closed. By contrast, the category of normal qcb-spaces is not cartesian closed: the space $\mathbb{N}^{(\mathbb{N}^{\mathbb{N}})}$ of Kleene-Kreisel continuous functionals of order 2 is not normal (see [Schröder 09]), even though $\mathbb{N}^{\mathbb{N}}$ and $\mathbb{N}$ are metrisable. An open question is whether the category of quasi-normal qcb-spaces endowed with an admissible effectively functionally closed representation is cartesian closed.

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