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Chainable and Circularly Chainable Co-r.e. Sets in Computable Metric Spaces

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Abstract: We investigate under what conditions a co-recursively enumerable set S in a computable metric space (X, d, α) is recursive. The topological properties of S play an important role in view of this task. We first study some properties of computable metric spaces such as the effective covering property. Then we examine co-r.e. sets with disconnected complement, and finally we focus on study of chainable and circularly chainable continua which are co-r.e. as subsets of X. We prove that, under some assumptions on X, each co-r.e. circularly chainable continuum which is not chainable must be recursive. This means, for example, that each co-r.e. set in $\mathbf{R}^n$ or in the Hilbert cube which has topological type of the Warsaw circle or the dyadic solenoid must be recursive. We also prove that for each chainable continuum S which is decomposable and each $\varepsilon > 0$ there exists a recursive subcontinuum of S which is $\varepsilon-$close to S.
Key Words: computable metric space, recursive set, co-r.e. set, chainable continuum, circularly chainable continuum, the effective covering property
Category: F.0, G.0

1 Introduction

It is known that there exists a computable function $f : \mathbf{R} \to \mathbf{R}$ which has zero-points, but none of them is recursive [Specker 1959]. However, it is also known that if a computable function $f : \mathbf{R}^n \to \mathbf{R}$, $n \geq 1$ has an isolated zero-point, then that point must be recursive.

From this we conclude the following: if $f : \mathbf{R} \to \mathbf{R}$ is a computable function such that $\mathbf{R} \setminus f^{-1}(\{0\})$ has finitely many components, at least two, then f has a recursive zero-point. Since $S \subseteq \mathbf{R}^n$ is of the form $f^{-1}(\{0\})$ for some computable function if and only if it is co-recursively enumerable, which means that $X \setminus S$ can be effectively covered by open balls, we have the following question: what can be said in general in view of recursive points of an co-r.e. set S in $\mathbf{R}^n$ whose complement is disconnected?

As we shall see, a co-r.e. set S must contain a recursive zero-point if its complement has finitely many components, at least two. It turns out that, under some additional assumptions, we have even more general result: S is recursive, which means that the distance function $d_S : \mathbf{R}^n \to \mathbf{R}$, $d_S(x) = d(x, S)$, $x \in \mathbf{R}^n$, is computable.

Each recursive set is co-recursively enumerable, while on the other hand there exist co-recursively enumerable sets which contain no recursive points, hence which are "far away from being recursive".

Having on mind that the implication

$$S \text{ co-recursively enumerable} \Rightarrow S \text{ recursive} \quad (1)$$

does not hold in general, we prove that under some topological assumptions (1) holds, not just in Euclidean space, but more generally in some computable metric spaces. Here we will need some additional assumptions on a computable metric space such as the effective covering property.

As a consequence we will get that (1) holds for $S \subseteq \mathbf{R}^n$ homeomorphic to a sphere of dimension $n-1$. This is just a special case of the result proved by Miller in [Miller 2002]: If $S \subseteq \mathbf{R}^n$ is homeomorphic to a sphere of any dimension, then (1) holds. Miller also proved that if $f : D \to \mathbf{R}^n$ is a continuous injection, where $D \subseteq \mathbf{R}^m$ is a closed ball, such that $f(D)$ and $f(\partial D)$ are co-recursively enumerable sets in $\mathbf{R}^n$, then $f(D)$ is a recursive set. In general, if $f(D)$ is co-recursively enumerable, then $f(D)$ need not be recursive, but the set of recursive points of $f(D)$ must be dense in $f(D)$. This is proved in [Miller 2002].

In particular, (1) holds if S is homeomorphic to a circle or if S is an arc with recursive endpoints. In this paper we study chainable (arc-like) continua and circularly chainable (circle-like) continua as co-r.e. subsets of a computable metric space (X, d, α) which has the effective covering property and compact closed balls, for example Euclidean space or the Hilbert cube. There are two main results which we prove: if S is a circularly chainable continuum which is not chainable, then (1) holds (Theorem 35); if S is a chainable continuum which is decomposable, then for each $\varepsilon > 0$ there exists a subcontinuum of S which is recursive and which is $\varepsilon-$close to S with respect to Hausdorff metric (Theorem 42). We also prove that each co-r.e. continuum chainable from a to b, where a and b are recursive points, is recursive (Theorem 36).

2 Basic techniques

Let $k, n \in \mathbf{N}$, $k, n \geq 1$. By a partially recursive function $f : S \to \mathbf{N}^n$, $S \subseteq \mathbf{N}^k$, we mean a function whose component functions $f_1, \dots, f_n : S \to \mathbf{N}$ are partially recursive. Of course, such a function will be called recursive if $S = \mathbf{N}^k$. In the following proposition we state some elementary facts.

Proposition 1. *(i) Let $T \subseteq \mathbf{N}^{k+n}$ be a recursively enumerable set. Then the set $S = \{x \in \mathbf{N}^k \mid \exists y \in \mathbf{N}^n \ (x, y) \in T\}$ is recursively enumerable. If $S_1 \subseteq \mathbf{N}^k$ and $S_2 \subseteq \mathbf{N}^n$ are r.e. sets such that for each $x \in S_1$ there exists $y \in S_2$ such that $(x, y) \in T$, then there exists a partially recursive function $f : S_1 \to \mathbf{N}^n$ such that $f(S_1) \subseteq S_2$ and $(x, f(x)) \in T$, $\forall x \in S_1$.*

(ii) If $S \subseteq \mathbf{N}^n$ is an r.e. set and $f : \mathbf{N}^k \to \mathbf{N}^n$ is a recursive function, then the set $\{x \in \mathbf{N}^k \mid f(x) \in S\}$ (i.e. $f^{-1}(S)$) is r.e.

One of the basic notions that we use in this paper is the notion of a recursive function $f : \mathbf{N}^k \to \mathbf{R}$, which generalizes the notion of a recursive sequence in $\mathbf{R}$: f is recursive if there exists a recursive rational approximation of f, i.e. a recursive function $F : \mathbf{N}^{k+1} \to \mathbf{Q}$ (which means $F(y) = (-1)^{c(y)}\frac{a(y)}{b(y)}$, where $a, b, c : \mathbf{N}^{k+1} \to \mathbf{N}$ are recursive functions, $b(y) \neq 0$) such that $|f(x) - F(x, i)| < 2^{-i}$, $\forall x \in \mathbf{N}^k$, $\forall i \in \mathbf{N}$.

In the following proposition we state some elementary facts about recursive functions $\mathbf{N}^k \to \mathbf{R}$.

Proposition 2. *(i) If $f, g : \mathbf{N}^k \to \mathbf{R}$ are recursive, then $f + g, f - g : \mathbf{N}^k \to \mathbf{R}$ are recursive.*

(ii) If $f : \mathbf{N}^k \to \mathbf{R}$ and $F : \mathbf{N}^{k+1} \to \mathbf{R}$ are functions such that F is recursive and $|f(x) - F(x, i)| < 2^{-i}$, $\forall x \in \mathbf{N}^k$, $\forall i \in \mathbf{N}$, then f is recursive.

(iii) If $f : \mathbf{N}^{k+1} \to \mathbf{R}$ and $\varphi : \mathbf{N}^k \to \mathbf{N}$ are recursive functions, then the functions $g, h : \mathbf{N}^k \to \mathbf{R}$ defined by $g(x) = \max_{0 \le i \le \varphi(x)} f(i, x)$, $h(x) = \min_{0 \le i \le \varphi(x)} f(i, x)$, $x \in \mathbf{N}^k$, are recursive.

(iv) If $f, g : \mathbf{N}^k \to \mathbf{R}$ is a recursive function, then the set $\{x \in \mathbf{N}^k \mid f(x) > g(x)\}$ is r.e.

We say that a function $\Phi : \mathbf{N}^k \to \mathcal{P}(\mathbf{N}^n)$ is **recursive** if the function $\overline{\Phi} : \mathbf{N}^{k+n} \to \mathbf{N}$ defined by

$$\overline{\Phi}(x, y) = \chi_{\Phi(x)}(y),$$

$x \in \mathbf{N}^k$, $y \in \mathbf{N}^n$ is recursive. Here $\mathcal{P}(\mathbf{N}^n)$ denotes the set of all subsets of $\mathbf{N}^n$, and $\chi_S : \mathbf{N}^n \to \mathbf{N}$ denotes the characteristic function of $S \subseteq \mathbf{N}^n$. A function $\Phi : \mathbf{N}^k \to \mathcal{P}(\mathbf{N}^n)$ is said to be **recursively bounded** if there exists a recursive function $\varphi : \mathbf{N}^k \to \mathbf{N}$ such that $\Phi(x) \subseteq \{0, \dots, \varphi(x)\}^n$, $\forall x \in \mathbf{N}^k$, where $\{0, \dots, \varphi(x)\}^n$ equals the set of all $(y_1, \dots, y_n) \in \mathbf{N}^n$ such that $\{y_1, \dots, y_n\} \subseteq \{0, \dots, \varphi(x)\}$.

We say that a function $\Phi : \mathbf{N}^k \to \mathcal{P}(\mathbf{N}^n)$ is r.r.b. if Φ is recursive and recursively bounded. The proof of the following proposition is straightforward.

Proposition 3. *(i) If $f : \mathbf{N}^k \to \mathbf{N}^n$ is a recursive function, then the function $\Phi : \mathbf{N}^k \to \mathcal{P}(\mathbf{N}^n)$, $\Phi(x) = \{f(x)\}$, $x \in \mathbf{N}^k$, is r.r.b.*

(ii) If $f : \mathbf{N}^l \to \mathbf{N}^k$ is a recursive and $\Phi : \mathbf{N}^k \to \mathcal{P}(\mathbf{N}^n)$ is r.r.b., then $\Phi \circ f : \mathbf{N}^l \to \mathcal{P}(\mathbf{N}^n)$ is r.r.b.

(iii) If $\Phi, \Psi : \mathbf{N}^k \to \mathcal{P}(\mathbf{N}^n)$ are r.r.b. functions, then $x \mapsto \Phi(x) \cup \Psi(x)$, $x \mapsto \Phi(x) \cap \Psi(x)$, $x \mapsto \Phi(x) \setminus \Psi(x)$, $x \in \mathbf{N}^k$ are r.r.b. functions.

(iv) If $\Phi : \mathbf{N}^k \to \mathcal{P}(\mathbf{N}^n)$ is an r.r.b. function, then the set $\{x \in \mathbf{N}^k \mid \Phi(x) = \emptyset\}$ is recursive.

(v) If $\Phi, \Psi : \mathbf{N}^k \to \mathcal{P}(\mathbf{N}^n)$ are r.r.b. functions, then the sets $\{x \in \mathbf{N}^k \mid \Phi(x) = \Psi(x)\}$, $\{x \in \mathbf{N}^k \mid \Phi(x) \subseteq \Psi(x)\}$ are recursive.

It is not hard to prove the following proposition.

Proposition 4. *Let $\Phi : \mathbf{N}^k \to \mathcal{P}(\mathbf{N}^n)$ and $\Psi : \mathbf{N}^{n+k} \to \mathcal{P}(\mathbf{N}^m)$ be r.r.b. functions. Let $\Lambda : \mathbf{N}^k \to \mathcal{P}(\mathbf{N}^m)$ be defined by*

$$\Lambda(x) = \bigcup_{z \in \Phi(x)} \Psi(z, x),$$

$x \in \mathbf{N}^k$. Then Λ is an r.r.b. function.

Example 1. If $\alpha, \beta : \mathbf{N}^k \to \mathbf{N}$ are recursive functions, then the function $\mathbf{N}^k \to \mathcal{P}(\mathbf{N}^{k+1})$, $x \mapsto \{(i, x) \mid i \in \mathbf{N},\ \alpha(x) \leq i \leq \beta(x)\}$, $x \in \mathbf{N}^k$ is obviously r.r.b. It follows from Proposition 3(i) and Proposition 4 that if $f : \mathbf{N}^{k+1} \to \mathbf{N}^n$ is a recursive function, then the function $\mathbf{N}^k \to \mathcal{P}(\mathbf{N}^n)$, $x \mapsto \{f(i, x) \mid \alpha(x) \leq i \leq \beta(x)\}$ is r.r.b.

Example 2. Let $g : \mathbf{N} \to \mathbf{N}^n$ be a recursive function. It follows from Example 1 that the function $\Phi : \mathbf{N} \to \mathcal{P}(\mathbf{N}^n)$, $\Phi(i) = \{g(0), \ldots, g(i)\}$, $i \in \mathbf{N}$, is r.r.b. We conclude the following: if $T \subseteq \mathbf{N}^n$ is r.e., then there exists an r.r.b. function $\Phi : \mathbf{N} \to \mathcal{P}(\mathbf{N}^n)$ such that $\Phi(i) \subseteq \Phi(i + 1)$, $\forall i \in \mathbf{N}$ and $T = \cup_{i \in \mathbf{N}} \Phi(i)$.

Using Example 2 and Proposition 3 it is easy to prove the following lemma.

Lemma 5. *Let $\Phi : \mathbf{N}^k \to \mathcal{P}(\mathbf{N}^n)$ be r.r.b. and let $T \subseteq \mathbf{N}^n$ be r.e. Then the set $S = \{x \in \mathbf{N}^k \mid \Phi(x) \subseteq T\}$ is r.e.*

3 Computable metric spaces

Let (X, d) be a metric space. For $x \in X$ and $r > 0$ we denote by $B(x, r)$ the open ball of radius r centered at x and by $\widehat{B}(x, r)$ the corresponding closed ball, i.e. $B(x, r) = \{y \in X \mid d(x, y) < r\}$, $\widehat{B}(x, r) = \{y \in X \mid d(x, y) \leq r\}$. If A and B are nonempty subsets of X, then $d(A, B)$ denotes the number $\inf\{d(a, b) \mid a \in A, b \in B\}$. By $\overline{A}$ we denote the closure of $A \subseteq X$. For all $x \in X$ and $r > 0$ we have $\overline{B(x, r)} \subseteq \widehat{B}(x, r)$, but equality $\overline{B(x, r)} = \widehat{B}(x, r)$ does not hold necessarily.

Let $\mathcal{H}$ be the set of all nonempty compact subsets of (X, d). The function $\rho : \mathcal{H} \times \mathcal{H} \to \mathbf{R}$ given by $\rho(A, B) = \max\{\sup\{d(a, B) \mid a \in A\}, \sup\{d(b, A) \mid b \in B\}$ is a metric on $\mathcal{H}$ and it is known under the name Hausdorff metric. It is easy to prove the following proposition.

Proposition 6. *Let $A, B \in \mathcal{H}$ and $\varepsilon > 0$. The following statements are equivalent:*

(i) $\rho(A, B) < \varepsilon$;

(ii) for each $a \in A$ there exists $b \in B$ such that $d(a, b) < \varepsilon$ and for each $b \in B$ there exists $a \in A$ such that $d(b, a) < \varepsilon$;

(iii) $|d(x, A) - d(x, B)| < \varepsilon$, for each $x \in X$.

A computable metric space is a tuple (X, d, α), where (X, d) is a metric space and $\alpha : \mathbf{N} \to X$ is a sequence dense in (X, d) such that the function $\mathbf{N}^2 \to \mathbf{R}$, $(i, j) \mapsto d(\alpha_i, \alpha_j)$ is recursive.

Let $q : \mathbf{N} \to \mathbf{Q}$ be some fixed recursive function whose image is $\mathbf{Q} \cap \langle 0, \infty\rangle$. Let $\tau_1, \tau_2 : \mathbf{N} \to \mathbf{N}$ be some fixed recursive functions such that $\{(\tau_1(i), \tau_2(i)) \mid i \in \mathbf{N}\} = \mathbf{N}^2$. We are going to use the following notation: $\langle i\rangle_1$ instead of $\tau_1(i)$ and $\langle i\rangle_2$ instead of $\tau_2(i)$.

For $i \in \mathbf{N}$ we define

$$I_i = B(\alpha_{\langle i\rangle_1}, q_{\langle i\rangle_2}),\ \widehat{I}_i = \widehat{B}(\alpha_{\langle i\rangle_1}, q_{\langle i\rangle_2}).$$

Note that $\overline{I_i} \subseteq \widehat{I}_i$ and that $\overline{I_i}$ need not be equal to $\widehat{I}_i$.

Let $\sigma : \mathbf{N}^2 \to \mathbf{N}$ and $\eta : \mathbf{N} \to \mathbf{N}$ be some fixed recursive functions with the following property: $\{(\sigma(j, 0), \ldots, \sigma(j, \eta(j))) \mid j \in \mathbf{N}\}$ is the set of all finite sequences in $\mathbf{N}$, i.e. the set $\{(a_0, \ldots, a_n) \mid n \in \mathbf{N},\ a_0, \ldots, a_n \in \mathbf{N}\}$. Such functions, for instance, can be defined using the Cantor pairing function. We are going to use the following notation: $(j)_i$ instead of $\sigma(j, i)$ and $\overline{j}$ instead of $\eta(j)$. Hence

$$\{((j)_0, \ldots, (j)_{\overline{j}}) \mid j \in \mathbf{N}\}$$

is the set of all finite sequences in $\mathbf{N}$.

For $j \in \mathbf{N}$ the set $\{(j)_i \mid 0 \le i \le \overline{j}\}$ will be denoted by $[j]$. By Example 1 the function $\mathbf{N} \to \mathcal{P}(\mathbf{N})$, $j \mapsto [j]$ is r.r.b. For $j \in \mathbf{N}$ we define

$$J_j = \bigcup_{i\in[j]} I_i,\quad \widehat{J}_j = \bigcup_{i\in[j]} \widehat{I}_i.$$

The sets J_j represent finite unions of "rational balls" and the sets $\widehat{J}_j$ finite unions of "closed rational balls."

Let $a, b \in \mathbf{N}$. We say that a and b **represent formally disjoint balls** if $d(\alpha_{\langle a\rangle_1}, \alpha_{\langle b\rangle_1}) > q_{\langle a\rangle_2} + q_{\langle b\rangle_2}$. Clearly, if a and b represent formally disjoint balls, then $I_a \cap I_b = \emptyset$.

Let $i, j \in \mathbf{N}$. We say that i and j **represent formally disjoint unions of balls** if a and b represents formally disjoint balls for all $a \in [i]$, $b \in [j]$. If i and j have this property, then clearly $J_i \cap J_j = \emptyset$.

Lemma 7. *If A and B are compact disjoint subsets of (X, d, α), then there exists $n, m \in \mathbf{N}$ such that $A \subseteq J_n$, $B \subseteq J_m$ and such that n and m represent formally disjoint unions of balls.*

Proof. Let $\lambda = d(A, B)$. Then $\lambda > 0$. Let $\mathcal{I} = \{i \in \mathbf{N} \mid q_{\langle i\rangle_2} < \frac{\lambda}{4}\}$. Then $\{I_i \mid i \in \mathcal{I}\}$ is a family of open sets which covers both A and B and therefore there exist $i_0, \ldots, i_v, j_0, \ldots, j_w \in \mathcal{I}$ such that the family $\{I_{i_0}, \ldots, I_{i_v}\}$ covers A, each of its members intersects A, the family $\{I_{j_0}, \ldots, I_{j_w}\}$ covers B and each of its members intersects B. Let $a \in \{i_0, \dots, i_v\}$, $b \in \{j_0, \dots, j_w\}$. There exist $x \in A \cap I_a$, $y \in B \cap I_b$. We have $\lambda \leq d(x, y) \leq d(x, \alpha_{\langle a\rangle_1}) + d(\alpha_{\langle a\rangle_1}, \alpha_{\langle b\rangle_1}) + d(\alpha_{\langle b\rangle_1}, y) < q_{\langle a\rangle_2} + d(\alpha_{\langle a\rangle_1}, \alpha_{\langle b\rangle_1}) + q_{\langle b\rangle_2} < \frac{\lambda}{4} + d(\alpha_{\langle a\rangle_1}, \alpha_{\langle b\rangle_1}) + \frac{\lambda}{4}$, hence $\lambda < \frac{\lambda}{2} + d(\alpha_{\langle a\rangle_1}, \alpha_{\langle b\rangle_1})$ which implies $d(\alpha_{\langle a\rangle_1}, \alpha_{\langle b\rangle_1}) > \frac{\lambda}{2} > q_{\langle a\rangle_2} + q_{\langle b\rangle_2}$. This means that a and b represent formally disjoint balls. Let $n, m \in \mathbf{N}$ be such that $[n] = \{i_0, \dots, i_v\}$, $[m] = \{j_0, \dots, j_w\}$. Then n and m are the desired numbers. □

Proposition 8. *Let $\Delta = \{(j, j') \in \mathbf{N}^2 \mid j$ and j' represent formally disjoint unions of balls$\}$. Then Δ is a recursively enumerable set.*

Proof. Let $S = \{(a, b) \in \mathbf{N}^2 \mid a$ and b represent formally disjoint balls$\}$. It is easy to conclude from Proposition 2 that S is r.e. Now $\Delta = \{(j, j') \mid (a, b) \in S,\ \forall a \in [j], \forall b \in [j']\}$, hence

$$\Delta = \{(j, j') \mid ((j)_i, (j')_{i'}) \in S,\ \forall i, i' \text{ such that } 0 \leq i \leq \overline{j},\ 0 \leq i' \leq \overline{j'}\}. \qquad (2)$$

We define $\Phi : \mathbf{N}^2 \to \mathcal{P}(\mathbf{N}^2)$, $\Psi : \mathbf{N}^4 \to \mathcal{P}(\mathbf{N}^2)$ by $\Phi(j, j') = \{(i, i') \mid 0 \leq i \leq \overline{j},\ 0 \leq i' \leq \overline{j'}\}$, $\Psi(i, i', j, j') = \{((j)_i, (j')_{i'})\}$. It is clear that Φ and Ψ are r.r.b. functions. Let $\Lambda : \mathbf{N}^2 \to \mathcal{P}(\mathbf{N}^2)$ be defined by $\Lambda(l, l') = \cup_{(i,i') \in \Phi(l,l')} \Psi(i, i', l, l')$. Then Λ is r.r.b. by Proposition 4. We have $\Lambda(l, l') = \{((j)_i, (j')_{i'}) \mid 0 \leq i \leq \overline{j},\ 0 \leq i' \leq \overline{j'}\}$ and it follows from (2) that $\Delta = \{(l, l') \mid \Lambda(l, l') \subseteq S\}$. By Lemma 5 Δ is recursively enumerable. □

The following lemmas are immediate consequences of Proposition 1(i), Proposition 2(iv) and the fact that $(I_a \cap I_b \neq \emptyset \Leftrightarrow \exists k \in \mathbf{N}\ \alpha_k \in I_a,\ \alpha_k \in I_b)$, $\forall a, b \in \mathbf{N}$.

Lemma 9. *The set $\{(k, c) \in \mathbf{N}^2 \mid \alpha_k \in I_c\}$ is r.e.*

Lemma 10. *The set $S = \{(a, b) \mid I_a \cap I_b = \emptyset\}$ is r.e.*

Let (X, d) be a metric space and $x_0, \dots, x_k \in X$, $r_0, \dots, r_k \in \langle 0, \infty\rangle$. The **formal diameter** associated to the finite sequence $(x_0, r_0), \dots, (x_k, r_k)$ is the number $D \in \mathbf{R}$ defined by

$$D = \max_{0 \leq v, w \leq k} d(p_v, p_w) + 2 \max_{0 \leq v \leq k} r_v.$$

Lemma 11. *Let (X, d) be a metric space.*

(i) If D is the formal diameter associated to the finite sequence $(x_0, r_0), \dots, (x_k, r_k)$, then $\operatorname{diam}(\widehat{B}(x_0, r_0) \cup \dots \cup \widehat{B}(x_k, r_k)) \leq D$.

(ii) Let K and U be subsets of (X, d) such that K is nonempty and compact, U is open and $K \subseteq U$. Let A be a dense subset of (X, d). Then for each $\varepsilon > 0$ there exist $k \in \mathbf{N}$ and $x_0, \dots x_k \in A$, $r_0, \dots, r_k \in \mathbf{Q}$ such that $K \subseteq \cup_{0 \leq i \leq k} B(x_i, r_i)$, $\cup_{0 \leq i \leq k} \widehat{B}(x_i, r_i) \subseteq U$ and $D < \operatorname{diam}(K) + \varepsilon$, where D is the formal diameter associated to $(x_0, r_0), \dots, (x_k, r_k)$.

Proof. (i) This follows from the fact that $\mathrm{diam}(\widehat{B}(a,t) \cup \widehat{B}(b,s)) \leq d(a,b)+t+s$, for all $a,b \in X$, $t,s \in \langle 0,\infty\rangle$.

(ii) Let $\lambda = d(K, X \setminus U)$ (if $U = X$ we put $\lambda = 1$). Compactness of K implies $\lambda > 0$. Let $r \in \mathbf{Q}$ be such that $0 < r < \min\{\frac{\varepsilon}{4}, \frac{\lambda}{2}\}$. The family of open sets $\{B(x,r) \mid x \in A\}$ covers X and since K is compact there exist $k \in \mathbf{N}$ and $x_0, \dots x_k \in A$ such that $K \subseteq \cup_{0 \leq i \leq k} B(x_i, r)$. We may assume that $B(x_i,r) \cap K \neq \emptyset$, $\forall i \in \{0,\dots,k\}$. This implies that $d(y,K) \leq 2r$ (hence $d(y,K) < \lambda$) if $i \in \{0,\dots,m\}$ and $y \in \widehat{B}(x_i,r)$. We conclude from this that $\cup_{0\leq i\leq k}\widehat{B}(x_i,r_i) \subseteq U$.

Let $i,j \in \{0,\dots,k\}$. Then there exist $a,b \in K$ such that $d(x_i,a) < r$, $d(x_j,b) < r$. It follows $d(x_i,x_j) < d(a,b) + 2r$ and therefore $d(x_i,x_j) + 2r < \mathrm{diam}(K)+4r < \mathrm{diam}(K)+\varepsilon$. Hence the formal diameter associated to $(x_0,r), \dots, (x_k,r)$ is less than $\mathrm{diam}(K)+\varepsilon$. □

Let $\mathrm{fdiam} : \mathbf{N} \to \mathbf{R}$ be the function defined in the following way. For $j \in \mathbf{N}$ the number $\mathrm{fdiam}(j)$ is the formal diameter associated to the finite sequence $(\alpha_{\langle (j)_0\rangle_1}, q_{\langle (j)_0\rangle_2}), \dots, (\alpha_{\langle (j)_{\overline{j}}\rangle_1}, q_{\langle (j)_{\overline{j}}\rangle_2})$. As a direct consequence of Lemma 11 and the definition of sets J_j and $\widehat{J}_j$ we have the following proposition.

Proposition 12. *Let (X,d,α) be a computable metric space.*

(i) For all $j \in \mathbf{N}$, $\mathrm{diam}(\widehat{J}_j) \leq \mathrm{fdiam}(j)$.

(ii) Let K and U be subsets of (X,d) such that K is nonempty and compact, U is open and $K \subseteq U$. Let $\varepsilon > 0$. Then there exists $j \in \mathbf{N}$ such that $K \subseteq J_j$, $\widehat{J}_j \subseteq U$ and $\mathrm{fdiam}(j) < \mathrm{diam}(K)+\varepsilon$.

Proposition 13. *The function $\mathrm{fdiam} : \mathbf{N} \to \mathbf{R}$ is recursive.*

Proof. It follows from the definition of fdiam that

$$\mathrm{fdiam}(j) = \max_{0\leq v\leq \overline{j}} \left(\max_{0\leq w\leq \overline{j}} d\left(\alpha_{\langle (j)_v\rangle_1}, \alpha_{\langle (j)_w\rangle_1}\right)\right) + 2 \max_{0\leq v\leq \overline{j}} q_{\langle (j)_v\rangle_2},$$

$\forall j \in \mathbf{N}$. Now Proposition 2 implies that fdiam is a recursive function. □

Let (X,d,α) be a computable metric space. We say that $x \in X$ is a **recursive point** in (X,d,α) if there exists a recursive function $f : \mathbf{N} \to \mathbf{N}$ such that $d(x, \alpha_{f(k)}) < 2^{-k}$, $\forall k \in \mathbf{N}$.

A closed subset S of (X,d) is said to be **recursively enumerable** in (X,d,α) if $\{i \in \mathbf{N} \mid S \cap I_i \neq \emptyset\}$ is an r.e. subset of $\mathbf{N}$.

A closed subset S is said to be **co-recursively enumerable** in (X,d,α) if $S = \emptyset$ or there exists a recursive function $f : \mathbf{N} \to \mathbf{N}$ such that $X \setminus S = \cup_{i\in\mathbf{N}} I_{f(i)}$. (It is easy to see that these definitions do not depend on functions τ_1, τ_2, q which are necessary in the definition of I_i). If S is a closed subset of (X,d) such that the set $\{i \in \mathbf{N} \mid S \cap \widehat{I}_i = \emptyset\}$ is r.e., then S is clearly co-recursively enumerable.

(Under some additional assumptions on (X, d, α) the converse is also true, see Lemma 17 or Corollary 3.14 in [Brattka and Presser 2003].)

We say that S is a **recursive** set in (X, d, α) if S is both recursively enumerable and co-recursively enumerable.

Let S be a nonempty closed subset of (X, d, α) such that the function $\mathbf{N} \to \mathbf{R}$, $i \mapsto d(\alpha_i, S)$ is recursive. Then S must be recursive. Namely, for $i \in \mathbf{N}$ we have $S \cap I_i \neq \emptyset \Leftrightarrow d(\alpha_{\langle i \rangle_1}, S) < q_{\langle i \rangle_2}$ and $S \cap \widehat{I}_i = \emptyset \Leftrightarrow d(\alpha_{\langle i \rangle_1}, S) > q_{\langle i \rangle_2}$ which, together with Proposition 2(iv), implies that S is recursively enumerable and co-recursively enumerable, hence S is recursive.

Lemma 14. *Let S be a compact subset of (X, d, α) with the property that there exists an r.r.b. function $\Phi : \mathbf{N} \to \mathcal{P}(\mathbf{N})$ such that $S \subseteq \cup_{j \in \Phi(k)} J_j$ and $S \cap J_j \neq \emptyset$, $\operatorname{diam}(J_j) < 2^{-k}$, $\forall j \in \Phi(k)$, $\forall k \in \mathbf{N}$. Then S is recursive.*

Proof. Assume $S \neq \emptyset$. It is obvious from the definition of J_j that there exists a recursive function $f : \mathbf{N} \to \mathbf{N}$ such that $\alpha_{f(j)} \in J_j$, $\forall j \in \mathbf{N}$. Let $k \in \mathbf{N}$. We have the following conclusion: for each $s \in S$ there exists $j \in \mathbf{N}$ such that $d(s, \alpha_{f(j)}) < 2^{-k}$ and for each $j \in \Phi(k)$ there exists $s \in S$ such that $d(s, \alpha_{f(j)}) < 2^{-k}$. It follows from Proposition 6 that

$$|d(\alpha_i, S) - d(\alpha_i, \{\alpha_{f(j)} \mid j \in \Phi(k)\})| < 2^{-k},$$

$\forall i \in \mathbf{N}$. Therefore, by Proposition 2(ii), it is enough to prove that the function $g : \mathbf{N}^2 \to \mathbf{R}$, $g(i, k) = d(\alpha_i, \{\alpha_{f(j)} \mid j \in \Phi(k)\})$ is recursive. By Proposition 3 there exists a recursive function $\varphi : \mathbf{N} \to \mathbf{N}$ such that $\Phi(k) = [\varphi(k)]$, $\forall k \in \mathbf{N}$. Hence $\varphi(k) = \{(\varphi(k))_0, \dots, (\varphi(k))_{\overline{\varphi(k)}}\}$ and this implies that $g(i, k) = \min_{0 \leq j \leq \overline{\varphi(k)}} d(\alpha_i, \alpha_{f((\varphi(k))_j)})$. The fact that g is recursive follows now from Proposition 2(iii). □

3.1 Effective covering property

A computable metric space (X, d, α) has the **effective covering property** if the set $\{(w, j) \in \mathbf{N}^2 \mid \widehat{I}_w \subseteq J_j\}$ is r.e. [Brattka and Presser 2003]. It is not hard to see that this definition does not depend on the choice of the functions $q, \tau_1, \tau_2, \sigma, \eta$ which are necessary in the definitions of sets I_w and J_j.

Proposition 15. *Let (X, d, α) be a computable metric space which has the effective covering property. Then the set $S = \{(i, j) \in \mathbf{N}^2 \mid \widehat{J}_i \subseteq J_j\}$ is r.e.*

Proof. If $i, j \in \mathbf{N}$, then $\widehat{J}_i \subseteq J_j \Leftrightarrow \widehat{I}_w \subseteq J_j$, $\forall w \in [i]$. Let $\Phi : \mathbf{N}^2 \to \mathcal{P}(\mathbf{N}^2)$ be given by $\Phi(i, j) = \{(w, j) \mid w \in [i]\}$. Then Φ is r.r.b. since $i \mapsto [i]$, $i \in \mathbf{N}$ is r.r.b. We have $(i, j) \in S \Leftrightarrow \Phi(i, j) \subseteq \{(n, m) \mid \widehat{I}_n \subseteq J_m\}$. By Lemma 5 S is r.e. □

We say that a metric space (X, d) **has compact closed balls** if $\widehat{B}(x, r)$ is a compact set for all $x \in X$, $r > 0$.

Proposition 16. *Let (X, d, α) be a computable metric space which has the effective covering property and compact closed balls. Then the set $S = \{(j, j') \in \mathbf{N}^2 \mid \widehat{J_j} \cap \widehat{J_{j'}} = \emptyset\}$ is recursively enumerable.*

Proof. Let $j, j' \in \mathbf{N}$. Since $\widehat{J_j}$ and $\widehat{J_{j'}}$ are compact sets, we have $\widehat{J_j} \cap \widehat{J_{j'}} = \emptyset$ if and only if there exist $l, l' \in \mathbf{N}$ such that $\widehat{J_j} \subseteq J_l$, $\widehat{J_{j'}} \subseteq J_{l'}$ and such that l and l' represent formally disjoint unions of balls (Lemma 7). By Proposition 8 and Proposition 15 S is r.e. □

Lemma 17. *Let (X, d, α) be a computable metric space which has the effective covering property and compact closed balls. Let S be a co-r.e. set in (X, d, α). Then the set $\{j \in \mathbf{N} \mid \widehat{J_j} \subseteq X \setminus S\}$ is r.e.*

Proof. Suppose $S \neq \emptyset$. Using Proposition 4 and Proposition 3(v), we conclude from the definition of a co-r.e. set that there exists a recursive function $\varphi : \mathbf{N} \to \mathbf{N}$ such that $X \setminus S = \bigcup_{i \in \mathbf{N}} J_{\varphi(i)}$ and $J_{\varphi(i)} \subseteq J_{\varphi(i+1)}$, $\forall i \in \mathbf{N}$. Let $j \in \mathbf{N}$. Then, since $\widehat{J_j}$ is compact, $\widehat{J_j} \subseteq X \setminus S \Leftrightarrow \exists i \in \mathbf{N}$ such that $\widehat{J_j} \subseteq J_{\varphi(i)}$. Now the claim of the lemma follows from Proposition 15 and Proposition 1. □

Let (X, d) be a metric space. Let $x_0, y_0 \in X$, $r_0, s_0 \in \langle 0, \infty\rangle$. We say that (y_0, s_0) is **formally contained** in (x_0, r_0) if $d(x_0, y_0) + s_0 < r_0$. If (y_0, s_0) is formally contained in (x_0, r_0), then clearly $B(y_0, s_0) \subseteq B(x_0, r_0)$.

Let (X, d, α) be a computable metric space. Let A be the set of all $(i, i') \in \mathbf{N}^2$ such that $(\alpha_{\langle i\rangle_1}, q_{\langle i\rangle_2})$ is formally contained in $(\alpha_{\langle i'\rangle_1}, q_{\langle i'\rangle_2})$. It follows from Proposition 2(iv) that A is r.e. Let

$$B = \{(i, j) \in \mathbf{N}^2 \mid (i, i') \in A \text{ for some } i' \in [j]\}. \tag{3}$$

For $i, j \in \mathbf{N}$ we have $(i, j) \in B \Leftrightarrow \exists k \in \mathbf{N}$ such that $0 \leq k \leq \overline{j}$, $(i, (j)_k) \in A$. Therefore B is r.e. Note that

$$(i, j) \in B \ \Rightarrow\ I_i \subseteq J_j, \tag{4}$$

$\forall i, j \in \mathbf{N}$.

Lemma 18. *Let (X, d) be a metric space which has compact closed balls. Let U be an open set and $x \in X$, $r > 0$ such that $\widehat{B}(x, r) \subseteq U$. Then there exists $r' > r$ such that $\widehat{B}(x, r') \subseteq U$.*

Proof. Suppose that such an r' does not exist. Then for each $n \in \mathbf{N}$ there exists $x_n \in \widehat{B}(x, r + 2^{-n})$ such that $x_n \notin U$. Since $x_n \in \widehat{B}(x, r+1)$, $\forall n \in \mathbf{N}$ and $\widehat{B}(x, r+1)$ is a compact set, there exists a subsequence (x_{n_i}) of (x_n) which converges to $a \in X$. Then $d(x, x_{n_i}) \to d(x, a)$ and since $d(x, x_{n_i}) \leq r + 2^{-n_i}$, $\forall i \in \mathbf{N}$, we have $d(x, a) \leq r$. It follows $a \in U$ and therefore $x_{n_i} \in U$ for some $i \in \mathbf{N}$. A contradiction. □

Lemma 19. *Let (X,d) be a metric space, $a_0,\dots,a_n \in X$, $r_0,\dots,r_n \in \langle 0,\infty\rangle$. Let K be a compact set in (X,d) such that $K \subseteq B(a_0,r_0)\cup\ldots\cup B(a_n,r_n)$. Then there exists $\lambda > 0$ such that for each $x \in K$ the pair (x,λ) if formally contained in (a_i,r_i) for some $i \in \{0,\dots,n\}$.*

Proof. Suppose that such λ does not exist. Then for each $n \in \mathbf{N}$ there exists $x_n \in K$ such that $(x_n, 2^{-n})$ is not formally contained in any (a_i,r_i), $i \in \{0,\dots,n\}$. Let (x_{n_k}) be a subsequence of (x_n) which converges to $\widetilde{x} \in K$. We have $\widetilde{x} \in B(a_i,r_i)$ for some $i \in \{0,\dots,n\}$. Let $k \in \mathbf{N}$. We have

$$d(a_i,x_{n_k}) + 2^{-n_k} \le d(a_i,\widetilde{x}) + d(\widetilde{x},x_{n_k}) + 2^{-n_k}.$$

Since $d(a_i,\widetilde{x}) < r_i$ and $\lim_k(d(\widetilde{x},x_{n_k}) + 2^{-n_k}) = 0$, there exists $k \in \mathbf{N}$ such that $d(a_i,\widetilde{x}) + d(\widetilde{x},x_{n_k}) + 2^{-n_k} < r_i$. This implies $d(a_i,x_{n_k}) + 2^{-n_k} < r_i$ which means that $(x_{n_k}, 2^{-n_k})$ is formally contained in (a_i,r_i). A contradiction. □

Lemma 20. *Let (X,d,α) be a computable metric space which has compact closed balls and such that there exists an r.r.b. function $\Phi : \mathbf{N}^2 \to \mathcal{P}(\mathbf{N})$ with the following property:*

$$\widehat{I}_i \subseteq \bigcup_{j\in\Phi(i,k)} I_j \text{ and } q_{\langle j\rangle_2} \le 2^{-k},\ \forall j \in \Phi(i,k),\ \forall i,k \in \mathbf{N}. \tag{5}$$

Then there exists an r.r.b. function $\Psi : \mathbf{N}^2 \to \mathcal{P}(\mathbf{N})$ which also satisfies property (5) and such that the following holds: if U is an open set in (X,d) and $i \in \mathbf{N}$ such that $\widehat{I}_i \subseteq U$, then there exists $k_0 \in \mathbf{N}$ such that $\bigcup_{j\in\Psi(i,k)} I_j \subseteq U$, $\forall k \ge k_0$.

Proof. Let $\Lambda : \mathbf{N}^2 \to \mathbf{R}$ be defined by

$$\Lambda(i,j) = d(\alpha_{\langle i\rangle_1}, \alpha_{\langle j\rangle_1}) - q_{\langle i\rangle_2} - q_{\langle j\rangle_2},$$

$i,j \in \mathbf{N}$. Note that $\Lambda(i,j) > 0$ implies $\widehat{I}_i \cap I_j = \emptyset$. Since Λ is a recursive function, there exists a recursive function $C : \mathbf{N}^3 \to \mathbf{N}$ such that for all $i,j,k \in \mathbf{N}$

$$C(i,j,k) = 0 \ \Rightarrow\ \Lambda(i,j) > 0,$$

$$C(i,j,k) = 1 \ \Rightarrow\ \Lambda(i,j) < 2^{-k}.$$

The existence of this function follows easily from the fact that $\{(i,j,k) \mid \Lambda(i,j) > 0\}$ and $\{(i,j,k) \mid \Lambda(i,j) < 2^{-k}\}$ are r.e. sets whose union is $\mathbf{N}^3$. Let $\Psi : \mathbf{N}^2 \to \mathcal{P}(\mathbf{N})$ be defined by

$$j \in \Psi(i,k) \ \Leftrightarrow\ j \in \Phi(k) \text{ and } C(i,j,k) = 1.$$

Clearly Ψ is r.r.b. Let $i,k \in \mathbf{N}$. Then $\Psi(i,k) \subseteq \Phi(i,k)$ and $\Lambda(i,j) > 0$, $\forall j \in \Phi(i,k) \setminus \Psi(i,k)$. Hence $\widehat{I}_i \cap I_j = \emptyset$, $\forall j \in \Phi(i,k) \setminus \Psi(i,k)$. It follows

$$\widehat{I}_i \subseteq \bigcup_{j\in\Psi(i,k)} I_j.$$

For each $j \in \Psi(i,k)$ we have $\Lambda(i,j) < 2^{-k}$ and therefore $d(\alpha_{\langle i\rangle_1}, \alpha_{\langle j\rangle_1}) < q_{\langle i\rangle_2} + q_{\langle j\rangle_2} + 2^{-k} \leq q_{\langle i\rangle_2} + 2 \cdot 2^{-k}$ which implies that for each $x \in I_j$

$$d(\alpha_{\langle i\rangle_1}, x) \leq d(\alpha_{\langle i\rangle_1}, \alpha_{\langle j\rangle_1}) + d(\alpha_{\langle j\rangle_1}, x) < (q_{\langle i\rangle_2} + 2 \cdot 2^{-k}) + 2^{-k} = q_{\langle i\rangle_2} + 3 \cdot 2^{-k}.$$

We conclude that $\bigcup_{j\in\Psi(i,k)} I_j \subseteq B(\alpha_{\langle i\rangle_1}, q_{\langle i\rangle_2} + 3 \cdot 2^{-k})$, $\forall i, k \in \mathbf{N}$. The claim of the lemma now follows from Lemma 18. □

Theorem 21. *Let (X, d, α) be a computable metric space which has compact closed balls and such that there exists an r.r.b function $\Phi : \mathbf{N}^2 \to \mathcal{P}(\mathbf{N})$ with property (5). Then (X, d, α) has the effective covering property.*

Proof. Let Ψ be the function of Lemma 20. Suppose $i, v \in \mathbf{N}$ and $\widehat{I_i} \subseteq J_v$. It follows immediately from Lemma 18 that there exist an open set U and a compact set K such that $\widehat{I_i} \subseteq U \subseteq K \subseteq J_v$. Let $k_0 \in \mathbf{N}$ be such that $\bigcup_{j\in\Psi(i,k)} I_j \subseteq U$, $\forall k \geq k_0$. By Lemma 19 there exists $\lambda > 0$ such that for each $x \in K$ the pair (x, λ) is formally contained in $(\alpha_{\langle w\rangle_1}, q_{\langle w\rangle_2})$ for some $w \in [v]$. Let $k \in \mathbf{N}$ be such that $k \geq k_0$, $2^{-k} < \lambda$. Then we have $\cup_{j\in\Psi(i,k)} I_j \subseteq U$ and therefore $\alpha_{\langle j\rangle_1} \in K$, $\forall j \in \Psi(i,k)$. For $j \in \Psi(i,k)$ this fact, together with $q_{\langle j\rangle_2} \leq 2^{-k} < \lambda$, implies that $(\alpha_{\langle j\rangle_1}, q_{\langle j\rangle_2})$ is formally contained in $(\alpha_{\langle w\rangle_1}, q_{\langle w\rangle_2})$ for some $w \in [v]$. It follows $(j, v) \in B$, where B is the set given by (3). Conclusion: if $i, v \in \mathbf{N}$ are such that $\widehat{I_i} \subseteq J_v$, then there exists $k \in \mathbf{N}$ such that $(j, v) \in B$, $\forall j \in \Psi(i,k)$.

Conversely, let $i, v, k \in \mathbf{N}$ be such that $(j, v) \in B$, $\forall j \in \Psi(i,k)$. By (4) $I_j \subseteq J_v$, $\forall j \in \Psi(i,k)$. Hence $\cup_{j\in\Psi(i,k)} I_j \subseteq J_v$ and since $\widehat{I_i} \subseteq \cup_{j\in\Psi(i,k)} I_j$, we have $\widehat{I_i} \subseteq J_v$.

We have proved the following:

$$\widehat{I_i} \subseteq J_v \iff \exists k \in \mathbf{N} \text{ such that } (j, v) \in B,\ \forall j \in \Psi(i,k). \tag{6}$$

Let $\Lambda : \mathbf{N}^3 \to \mathcal{P}(\mathbf{N}^2)$ be defined by $\Lambda(i,k,v) = \{(j,v) \mid j \in \Psi(i,k)\}$. It is easy to conclude from Proposition 4 that Λ is an r.r.b. function. Now (6) implies $\widehat{I_i} \subseteq J_v \Leftrightarrow \exists k \in \mathbf{N}$ such that $\Lambda(i,k,v) \subseteq B$. It follows from Lemma 5 that $\{(i,v) \mid \widehat{I_i} \subseteq J_v\}$ is an r.e. set. Hence (X, d, α) has the effective covering property. □

Let (X, d, α) be a computable metric space. A sequence (x_i) in X is said to be **recursive** if there exists a recursive function $f : \mathbf{N}^2 \to \mathbf{N}$ such that $d(x_i, \alpha_{f(i,k)}) < 2^{-k}$, for each $i, k \in \mathbf{N}$.

Corollary 22. *Let (X, d, α) be a computable metric space, $a \in X$ a recursive point, (x_i) a recursive sequence in X and $F : \mathbf{N}^2 \to \mathbf{N}$ a recursive function such that*

$$B(a, M) \subseteq \bigcup_{0\leq j\leq F(M,k)} B(x_j, 2^{-k}),$$

$\forall M, k \in \mathbf{N}$. *Then (X, d, α) has the effective covering property.*

Proof. Since a is recursive, there exist a recursive function $f : \mathbf{N} \to \mathbf{N}$ such that $\widehat{I}_i \subseteq B(a, f(i))$, $\forall i \in \mathbf{N}$ and since (x_i) is recursive, there exists a recursive function $g : \mathbf{N}^2 \to \mathbf{N}$ such that $B(x_j, 2^{-(k+1)}) \subseteq B(\alpha_{g(j,k)}, 2^{-k})$, $\forall j, k \in \mathbf{N}$. Then $\widehat{I}_i \subseteq \bigcup_{0 \le j \le F(f(i),k+1)} B(\alpha_{g(j,k)}, 2^{-k})$, $\forall i, k \in \mathbf{N}$. It follows easily from this that there exists an r.r.b. function $\Phi : \mathbf{N}^2 \to \mathcal{P}(\mathbf{N})$ with property (5). By Theorem 21 (X, d, α) has the effective covering property. □

Example 3. Let d be Euclidean metric on $\mathbf{R}$. It is easy to find a recursive function $r : \mathbf{N} \to \mathbf{Q}$ and a recursive function $F : \mathbf{N}^2 \to \mathbf{N}$ such that $[-M, M] \subseteq \cup_{0 \le i \le F(M,k)} B(r_i, 2^{-k})$, $\forall M, k \in \mathbf{N}$. Therefore, if $\alpha : \mathbf{N} \to \mathbf{Q}$ is a recursive surjection, $(\mathbf{R}, d, \alpha)$ is a computable metric space with the effective covering property. Similarly, if $n \ge 1$, d Euclidean metric on $\mathbf{R}^n$ and $\alpha : \mathbf{N} \to \mathbf{Q}^n$ a recursive surjection, then $(\mathbf{R}^n, d, \alpha)$ is a computable metric space with the effective covering property.

Let I^∞ be the set of all sequences (x_i) of real numbers such that $x_i \in [0, 1]$, $\forall i \in \mathbf{N}$. We have the metric d on I^∞ defined by $d((x_i), (y_i)) = \sum_{i=0}^{\infty} \frac{1}{2^i} |x_i - y_i|$. It is known that this metric induces topology which coincides with the product topology on I^∞. The space I^∞ is called the Hilbert cube. As the product of compact spaces, the Hilbert cube is compact.

Let $r : \mathbf{N} \to \mathbf{Q}$ be some recursive function such that $r(\mathbf{N}) = [0, 1] \cap \mathbf{Q}$. Let $\alpha : \mathbf{N} \to I^\infty$ be defined by $\alpha(i) = (r_{(i)_0}, \ldots, r_{(i)_{\overline{i}}}, 0, 0, \ldots)$, $i \in \mathbf{N}$. Then (I^∞, d, α) is a computable metric space.

It follows from Example 3 that there exists a recursive function $f : \mathbf{N} \to \mathbf{N}$ such that $[0, 1] \subseteq \cup_{0 \le i \le f(k)} B(r_i, 2^{-k})$, $\forall k \in \mathbf{N}$. Let $F : \mathbf{N}^2 \to \mathbf{N}$ be a recursive function such that for all $l, k \in \mathbf{N}$ the set $\{(i_0, \ldots, i_k) \in \mathbf{N}^{k+1} \mid i_0 \le l, \ldots, i_k \le l\}$ is contained in the set $\{((i)_0, \ldots, (i)_k) \mid 0 \le i \le F(l, k),\ \overline{i} = k\}$.

Let $k \in \mathbf{N}$ and $(x_i) \in I^\infty$. For each $i \in \{0, \ldots, k\}$ let j_i be such that $j_i \le f(k)$ and $|x_i - r_{j_i}| < 2^{-k}$. Note that $(r_{j_0}, \ldots, r_{j_k}, 0, 0, \ldots) = \alpha(n)$, where $n \in \mathbf{N}$, $n \le F(f(k), k)$. The distance between $(x_0, \ldots, x_k, 0, 0, \ldots)$ and $(r_{j_0}, \ldots, r_{j_k}, 0, 0, \ldots)$ is less than $2 \cdot 2^{-k}$ and, on the other hand , the distance between (x_i) and $(x_0, \ldots, x_k, 0, 0, \ldots)$ is less than 2^{-k}. Therefore $d(\alpha(n), (x_i)) < 3 \cdot 2^{-k}$. We have proved the following:

$$I^\infty \subseteq \bigcup_{0 \le n \le F(f(k),k)} B(\alpha_n, 3 \cdot 2^{-k}),$$

$\forall k \in \mathbf{N}$. By Corollary 22 (I^∞, d, α) has the effective covering property.

4 Co-r.e. sets with disconnected complement

In this section we examine co-r.e. sets whose complements are disconnected. In Theorem 26 we give some conditions under which such a set contains a recursive point and some conditions under which such a set is recursive.

Let (X,d,α) be a computable metric space, $U \subseteq X$ and $a,b \in X$. Let $i_0,\dots,i_m \in \mathbf{N}$. We say that the finite sequence $i_0,\dots,i_m$ **connects** a and b in U if $a \in I_{i_0}$, $b \in I_{i_m}$, $I_{i_k} \cap I_{i_{k+1}} \neq \emptyset$, $\forall k \in \{0,\dots,m-1\}$ and $\widehat{I}_{i_0} \cup \ldots \cup \widehat{I}_{i_m} \subseteq U$.

Lemma 23. *Let (X,d,α) be a computable metric space and let U be an open and connected set in (X,d). Let $\varepsilon > 0$. Then for each $a,b \in U$ there exists a finite sequence $i_0,\dots,i_m$ in $\mathbf{N}$ which connects a and b in U and such that $q_{\langle i_0\rangle_2},\dots,q_{\langle i_m\rangle_2} < \varepsilon$.*

Proof. Let $\sim$ be the relation on U defined by $x \sim y \Leftrightarrow$ there exists a finite sequence $i_0,\dots,i_m$ in $\mathbf{N}$ which connects a and b in U and such that $q_{\langle i_0\rangle_2},\dots,q_{\langle i_m\rangle_2} < \varepsilon$. Then $\sim$ is clearly an equivalence relation on U and for each $x \in U$ the set $V_x = \{y \in U \mid x \sim y\}$ is open. Since $\{V_x \mid x \in U\}$ is a partition of U and U is connected, we have $V_x = U$, $\forall x \in U$ and the claim of the lemma follows. $\square$

If (X,d,α) is a computable metric space and U an open set in (X,d), let

$$\Delta^U = \{(a,b) \in \mathbf{N}^2 \mid \alpha_a \text{ and } \alpha_b \text{ lie in the same component of } U\}.$$

Proposition 24. *Let (X,d,α) be a computable metric space which has compact and connected closed balls and the effective covering property. Let S be a co-r.e. set in this space. Then $\Delta^{X\setminus S}$ is an r.e. set.*

Proof. Let

$$\widetilde{\Delta} = \{(a,b,j) \in \mathbf{N}^3 \mid (j)_0,\dots,(j)_{\overline{j}} \text{ connects } \alpha_a \text{ and } \alpha_b \text{ in } X \setminus S\}.$$

We claim that for all $a,b \in \mathbf{N}$ the following equivalence holds:

$$(a,b) \in \Delta^{X\setminus S} \;\Leftrightarrow\; \exists j\ (a,b,j) \in \widetilde{\Delta}. \tag{7}$$

Let $a,b \in \mathbf{N}$ and suppose α_a and α_b lie in the same component U of $X \setminus S$. If $x \in U$, then $\widehat{B}(x,r) \subseteq X \setminus S$ for some $r > 0$ and since $\widehat{B}(x,r)$ is connected we have $\widehat{B}(x,r) \subseteq U$. It follows that U is an open set. By Lemma 23 there exists a finite sequence $i_0,\dots,i_m$ which connects α_a and α_b in U. We conclude that there exists $j \in \mathbf{N}$ such that $(a,b,j) \in \widetilde{\Delta}$.

Conversely, suppose $(a,b,j) \in \widetilde{\Delta}$, $a,b,j \in \mathbf{N}$. It follows that there exists a finite sequence $B_0,\dots,B_m$ of closed balls such that $\alpha_a \in B_0$, $\alpha_b \in B_m$, $B_i \cap B_{i+1} \neq \emptyset$, $\forall i \in \{0,\dots,m-1\}$ and $B_0 \cup \ldots \cup B_m \subseteq X \setminus S$. In general, the union of two connected sets which have nonempty intersection is connected. Therefore $B_0 \cup \ldots \cup B_m$ is a connected set and since it lies in $X \setminus S$, it must lie in some component U of $X \setminus S$. It follows $\alpha_a, \alpha_b \in U$. Hence (7) holds.

In order to prove that $\Delta^{X\setminus S}$ is r.e., it is now sufficient to prove that $\widetilde{\Delta}$ is r.e. Let $\Omega = \{(i_1,i_2) \in \mathbf{N}^2 \mid I_{i_1} \cap I_{i_2} \neq \emptyset\}$. Then Ω is r.e. by Lemma 10. Let

$\Phi : \mathbf{N} \to \mathcal{P}(\mathbf{N}^2)$ be defined by $\Phi(j) = \{(i, i+1) \mid 0 \le i < \overline{j}\}$ and $\Psi : \mathbf{N}^3 \to \mathcal{P}(\mathbf{N}^2)$ by $\Psi(i_1, i_2, j) = \{((j)_{i_1}, (j)_{i_2})\}$. These are clearly r.r.b. functions. If we apply Proposition 4 to Φ and Ψ we get that the function $\Lambda : \mathbf{N} \to \mathcal{P}(\mathbf{N}^2)$,

$$\Lambda(j) = \{((j)_i, (j)_{i+1}) \mid 0 \le i < \overline{j}\}$$

is r.r.b. We have

$$\widetilde{\Delta} = \{(a, b, j) \mid \alpha_a \in I_{(j)_0},\ \alpha_b \in I_{(j)_{\overline{j}}},\ \Lambda(j) \subseteq \Omega,\ \widehat{J}_j \subseteq X \setminus S\}.$$

The fact that $\widetilde{\Delta}$ is r.e. follows now from Lemma 9, Lemma 5 and Lemma 17. □

If (X, d, α) is a computable metric space and U an open set in (X, d), let

$$\Gamma^U = \{(a, b) \in \mathbf{N}^2 \mid \alpha_a \text{ and } \alpha_b \text{ lie in different components of } U\}.$$

Lemma 25. *Let $T \subseteq \mathbf{N}^{2k}$ and $S \subseteq \mathbf{N}^k$ be recursively enumerable sets such that for each $x \in S$ there exists $y \in S$ such that $(x, y) \in T$. Then for each $s \in S$ there exists a recursive function $f : \mathbf{N} \to \mathbf{N}^k$ such that $f(\mathbf{N}) \subseteq S$, $f(0) = s$ and $(f(i), f(i+1)) \in T$, $\forall i \in \mathbf{N}$.*

Proof. By Proposition 1(i) there exists a partially recursive function $h : S \to \mathbf{N}^k$ such that $h(S) \subseteq S$ and $(x, h(x)) \in T$, $\forall x \in S$. Let $s \in S$. Let $f : \mathbf{N} \to \mathbf{N}^k$ be defined by $f(0) = s$, $f(i+1) = h(f(i))$ (primitive recursion). Then $f : \mathbf{N} \to \mathbf{N}^k$ is the desired function. □

Theorem 26. *Let (X, d, α) be a computable metric space which has compact and connected closed balls and the effective covering property. Let S be a co-r.e. set in (X, d, α) such that there exists an r.e. subset A of $\mathbf{N}$ with the property that for each component C of $X \setminus S$ there exists unique $a \in A$ such that $\alpha_a \in C$. (Such a set A exists, for example, if $X \setminus S$ has finitely many components.) Then*

(i) $\Gamma^{X \setminus S}$ is r.e.;

(ii) if $x_0 \in X$, $r_0 > 0$ are such that $B(x_0, r_0)$ intersects two different components of $X \setminus S$, then $B(x_0, r_0) \cap S$ contains a recursive point;

(iii) if each point $x \in S$ lies in the boundary of at least two different components of $X \setminus S$, then S is a recursive set in (X, d, α).

Proof. (i) If $A = \emptyset$, then $X \setminus S = \emptyset$ and $\Gamma^{X \setminus S} = \emptyset$. Suppose $A \neq \emptyset$. Let $f : \mathbf{N} \to \mathbf{N}$ be a recursive function such that $A = f(\mathbf{N})$. Let $a, b \in \mathbf{N}$. Then

$$(a, b) \in \Gamma^{X \setminus S} \Leftrightarrow \exists i, j \in \mathbf{N}\ \ f(i) \neq f(j),\ (a, f(i)) \in \Delta^{X \setminus S},\ (b, f(j)) \in \Delta^{X \setminus S}.$$

Statement (i) now follows from Proposition 24.

Claim 1 Let $x \in X$ and $r > 0$. Then $B(x, r)$ is a connected set and if there exist $a, b \in \mathbf{N}$ are such that $\alpha_a, \alpha_b \in B(x, r)$, $(a, b) \in \Gamma^{X \setminus S}$, then $B(x, r) \cap S \neq \emptyset$.

We have $B(x,r) = \cup_{0<s<r}\widehat{B}(x,s)$ and therefore $B(x,r)$, as the union of connected sets with nonempty intersection, is connected. If $\alpha_a,\ \alpha_b \in B(x,r)$, $(a,b) \in \Gamma^{X\setminus S}$ for some $a,b \in \mathbf{N}$, then $B(x,r)$ intersects two different components of $X \setminus S$ and since it is connected, it cannot be contained in $X \setminus S$, hence $B(x,r) \cap S \neq \emptyset$.

Claim 2 Let $x \in X$, $r > 0$ and $a,b \in B(x,r)$ such that a and b lie in different components of $X \setminus S$. Suppose the interior of the set $B(x,r) \cap S$ is empty, i.e. $\mathrm{Int}(B(x,r) \cap S) = \emptyset$. Then for each $\varepsilon > 0$ there exists $i \in \mathbf{N}$ such that $\widehat{I}_i \subseteq B(x,r)$, $q_{\langle i\rangle_2} < \varepsilon$ and I_i intersects two different components of $X \setminus S$.

By Lemma 23 there exists a finite sequence $i_0,\dots,i_m$ which connects a and b in $B(x,r)$ and $q_{\langle i_0\rangle_2},\dots,q_{\langle i_m\rangle_2} < \varepsilon$. Let C be the component of $X \setminus S$ which contains a and let i_p be the first number in the sequence $i_0,\dots,i_m$ such that I_{i_p} intersects some component of $X \setminus S$ different from C. If $p = 0$, then I_{i_0} intersects two different components of $X \setminus S$. If $p > 0$, then $I_{i_{p-1}}$ must lie in $C \cup S$. However, since $I_{i_{p-1}} \cap I_{i_p}$ is open and nonempty and $\mathrm{Int}(B(x,r) \cap S) = \emptyset$, $I_{i_{p-1}} \cap I_{i_p}$ cannot be contained in S, hence it must intersect C. This implies $I_{i_p} \cap C \neq \emptyset$, hence I_{i_p} intersects two different components of $X \setminus S$. This completes the proof of Claim 2.

(ii) If $\mathrm{Int}(B(x_0,r_0) \cap S) \neq \emptyset$, then there exists $a \in \mathbf{N}$ such that $\alpha_a \in B(x_0,r_0) \cap S$ and this is the desired recursive point. Suppose $\mathrm{Int}(B(x_0,r_0) \cap S) = \emptyset$. By Claim 2 there exists $P \in \mathbf{N}$ such that $\widehat{I}_P \subseteq B(x_0,y_0)$ and I_P intersects two different components of $X \setminus S$. Let Ω be the set of all $i \in \mathbf{N}$ such that I_i intersects two different components of $X \setminus S$, $\widehat{I}_i \subseteq I_P$ and $q_{\langle i\rangle_2} < 1$. Then

$$i \in \Omega \;\Leftrightarrow\; (\exists a,b \in \mathbf{N}\ \alpha_a,\alpha_b \in I_i,\ (a,b) \in \Gamma^{X\setminus S}) \text{ and } \widehat{I}_i \subseteq I_P,\ q_{\langle i\rangle_2} < 1.$$

It follows from (i) that Ω is r.e. (note that $\{(i,j) \mid \widehat{I}_i \subseteq I_j\}$ is r.e. which follows from the obvious fact that there exists a recursive function $f : \mathbf{N} \to \mathbf{N}$ such that $I_j = J_{f(j)}$, $\forall j \in \mathbf{N}$). Let

$$\widetilde{\Omega} = \{(i,i') \in \mathbf{N}^2 \mid \widehat{I}_{i'} \subseteq I_i,\ q_{\langle i'\rangle_2} < \frac{1}{2}q_{\langle i\rangle_2}\}.$$

Then $\widetilde{\Omega}$ is r.e. By Claim 2 for each $i \in \Omega$ there exists $i' \in \Omega$ such that $(i,i') \in \widetilde{\Omega}$. By Lemma 25 there exists a recursive sequence $(i_k)_{k\in\mathbf{N}}$ such that $i_k \in \Omega$, $(i_k,i_{k+1}) \in \widetilde{\Omega}$, $\forall k \in \mathbf{N}$. By Claim 1

$$I_{i_k} \cap S \neq \emptyset,\quad \forall k \in \mathbf{N}. \tag{8}$$

We have $\widehat{I}_{i_{k+1}} \subseteq I_{i_k}$, $q_{\langle i_{k+1}\rangle_2} < \frac{1}{2}q_{\langle i_k\rangle_2}$, $\forall k \in \mathbf{N}$. It follows $\overline{I_{i_k}} \subseteq I_{i_k}$, $q_{\langle i_k\rangle_2} < 2^{-k}$, $\forall k \in \mathbf{N}$. We conclude that $\bigcap_{k\in\mathbf{N}} \overline{I_{i_k}} = \{\widetilde{x}\}$, where $\widetilde{x} \in X$. Since $d(\widetilde{x},\alpha_{\langle i_k\rangle_1}) < q_{\langle i_k\rangle_2} < 2^{-k}$, $\forall k \in \mathbf{N}$, $\widetilde{x}$ is a recursive point and (8) implies $\widetilde{x} \in \overline{S}$, hence $\widetilde{x} \in S$.

(iii) Let $i \in \mathbf{N}$. If $I_i \cap S \neq \emptyset$, then there exist $a,b \in \mathbf{N}$ such that $\alpha_a,\alpha_b \in I_i$, $(a,b) \in \Gamma^{X\setminus S}$. Now, by Claim 1, we have

$$I_i \cap S \neq \emptyset \;\Leftrightarrow\; \exists a,b \in \mathbf{N},\ \alpha_a,\ \alpha_b \in I_i,\ (a,b) \in \Gamma^{X\setminus S}.$$

This implies that S is a recursively enumerable set in (X, d, α). Therefore S is recursive. $\square$

Corollary 27. *Let S be a co-r.e. set in Euclidean space or in the Hilbert cube such that the complement of S has finitely many components, at least two. Then S contains a recursive point. If each point of S lies in the boundary of at least two different components of the complement of S, then S is recursive.*

Example 4. Let $S \subseteq \mathbf{R}$ be a nonempty co-r.e. set which does not contain any recursive number. We may suppose $S \subseteq \langle 0, 1\rangle$. Since S cannot contain an interval or an isolated point, $\mathbf{R}\backslash S$ has infinitely many components. Let $n \geq 2$ and $T = \{(t, 0, \dots, 0) \in \mathbf{R}^n \mid t \in S\}$, $T' = \{(x_1, x_2, \dots, x_n) \mid x_1 \in S,\ x_2, \dots, x_n \in \mathbf{R}\}$. Then T and T' are co-r.e. sets, $\mathbf{R}^n \setminus T$ is connected, $\mathbf{R}^n \setminus T'$ has infinitely many components and neither T nor T' contains a recursive point. On the other hand, let $T'' = T \cup \{x \in \mathbf{R}^n \mid \|x\| = 1\}$. It is not hard to check that T'', as the union of two co-r.e. sets, is a co-r.e. set. We have that $\mathbf{R}^n \setminus T''$ has precisely two components and it is easy to conclude that T'' is not recursive.

Let S be a co-r.e. subset of $\mathbf{R}^n$ which is, as a subspace of $\mathbf{R}^n$, a topological circle, i.e. a space homeomorphic to the unit circle $S^1 = \{x \in \mathbf{R}^2 \mid \|x\| = 1\}$. In the case $n = 2$ the Jordan curve theorem implies that $\mathbf{R}^2 \setminus S$ has precisely two components of which S is the common boundary and therefore, by Theorem 26, S is recursive. The property that the complement of S is disconnected is crucial in the proof of Theorem 26: it allows us to "approach" S from different sides and in that way we can effectively "locate" S. This is a situation similar to the one in which we have two recursive sequences of real numbers (x_i) and (y_i) and a number α as their common limit such that $x_i \leq \alpha \leq y_i$, $\forall i \in \mathbf{N}$. Then α must be a recursive number (in contrast to the fact that the limit of a recursive sequence need not be a recursive number). However, if $n > 2$, then $\mathbf{R}^n \setminus S$ is connected and therefore the technique used in the proof of Theorem 26 cannot be applied in this case. Nevertheless, there are some topological properties of the complement $\mathbf{R}^n \setminus S$ (which are in connection to certain homology groups) which make it possible to prove that S is recursive even when S is homeomorphic to sphere of any dimension [Miller 2002].

In the next section we will see that there is a class of spaces, more general than the class of topological circles, such that any co-r.e. set which belongs to that class must be recursive, not just in $\mathbf{R}^n$, but in any computable metric space which has the effective covering property and compact closed balls.

5 Co-r.e. chainable and circularly chainable continua

Let X be a metric space and $\mathcal{C} = (C_0, \dots, C_m)$ a finite sequence of subsets of X. By $\bigcup \mathcal{C}$ we denote the union $C_0 \cup \ldots \cup C_m$ and we say that $\mathcal{C}$ covers S,

$S \subseteq X$, if $S \subseteq \bigcup \mathcal{C}$. If C_i is nonempty for each $i \in \{0,\dots,m\}$, then we define $\mathrm{mesh}(\mathcal{C}) = \max_{0\leq i\leq m} \mathrm{diam}(C_i)$. We say that $\mathcal{C}$ is a finite sequence of $\varepsilon-$bounded sets, where $\varepsilon \in \mathbf{R}$, if $\mathrm{mesh}(\mathcal{C}) < \varepsilon$.

A finite sequence $C_0,\dots,C_m$ of nonempty open subsets of X is said to be a **chain** in X if $C_i \cap C_j \neq \emptyset \Leftrightarrow |i-j| \leq 1$, $\forall i,j \in \{0,\dots,m\}$ [Nadler 1992]. We say that C_i is a link of the chain $C_0,\dots,C_m$, $i \in \{0,\dots,m\}$.

That a chain $C_0,\dots,C_m$ in X covers S, $S \subseteq X$, means that it covers S as a finite sequence of sets, and that is an $\varepsilon-$chain, $\varepsilon \in \mathbf{R}$, means that it is a finite sequence of $\varepsilon-$bounded sets.

We say that X is a **chainable continuum** if X is a continuum (compact and connected metric space) such that for each $\varepsilon > 0$ there exists an ε-chain $C_0,\dots,C_m$ in X which covers X. Note that the sets $C_0,\dots,C_m$ in this definition need not be connected.

A finite sequence $C_0,\dots,C_m$ of nonempty open subsets of X is said to be a **circular chain** in X if $C_i \cap C_j \neq \emptyset \Leftrightarrow (|i-j| \leq 1$ or $\{i,j\} = \{0,m\})$. A continuum X is said to be **circularly chainable** if for each $\varepsilon > 0$ there exists a ε-circular chain in X which covers X [cf. Burgess 1959].

We say that a finite sequence $C_0,\dots,C_m$ of nonempty subsets of X is a **quasi-chain** if $C_i \cap C_j = \emptyset$ for all $i,j \in \{0,\dots,m\}$ such that $|i-j| > 1$. Similarly, we say that a finite sequence $C_0,\dots,C_m$ of nonempty subsets of X is a **circular quasi-chain** if $C_i \cap C_j = \emptyset$ for all $i,j \in \{0,\dots,m\}$ such that $|i-j| > 1$, $\{i,j\} \neq \{0,m\}$. If $\mathcal{C} = (C_0,\dots,C_m)$ is a (circular) quasi-chain and each of the sets $C_0,\dots,C_m$ is open, then we say that $\mathcal{C}$ is an open (circular) quasi-chain. Note the following: if $\mathcal{C} = (C_0,\dots,C_m)$ is an open quasi-chain, then $\mathcal{C}$ is not a chain if and only if there exists $i \in \{0,\dots,m-1\}$ such that $C_i \cap C_{i+1} = \emptyset$.

Lemma 28. *(i) Let* (X,d) *be a metric space,* S *compact subset of* (X,d) *and* $C_0,\dots,C_m$ *a finite sequence of open sets which covers* S*. Then there exist a finite sequence of open sets* $A_0,\dots,A_m$ *which covers* S *and such that* $\overline{A_i} \subseteq C_i$, $\forall i \in \{0,\dots,m\}$.

(ii) Let (X,d,α) *be a computable metric space,* S *compact subset of* (X,d), $r \in \mathbf{R}$ *and* $C_0,\dots,C_m$ *a finite sequence of* $r-$*bounded open sets which covers* S*. Then there exist* $j_0,\dots,j_m \in \mathbf{N}$ *such that the finite sequence of sets* $J_{j_0},\dots,J_{j_m}$ *covers* S, $\widehat{J_{j_i}} \subseteq C_i$ *and* $\mathrm{fdiam}(j_i) < r$, $\forall i \in \{0,\dots,m\}$.

Proof. We prove (ii), (i) is similar. Let $i \in \{0,\dots,m\}$. Let

$$K = S \setminus (C_0 \cup \ldots \cup C_{i-1} \cup C_{i+1} \cup \ldots \cup C_m).$$

Suppose $K \neq \emptyset$. Clearly K is closed and since it is a subset of S it must be compact. We have $K \subseteq C_i$. This implies $\mathrm{diam}(K) < r$. Now we conclude from Proposition 12 that there exists $j \in \mathbf{N}$ such that $K \subseteq J_j$, $\widehat{J_j} \subseteq C_i$ and $\mathrm{fdiam}(j) <$

r. It follows that $C_0, \ldots, C_{i-1}, J_j, C_{i+1}, \ldots, C_m$ covers S, $\widehat{J_j} \subseteq C_i$ and $\mathrm{fdiam}(j) < r$. The same conclusion we get in the case $K = \emptyset$: it is enough to take $j \in \mathbf{N}$ such that $\widehat{J_j} \subseteq C_i$ and $\mathrm{fdiam}(j) < r$ (note that $C_i \neq \emptyset$ by the definition of a finite sequence of $r-$bounded sets).

This means that there exists $j_0 \in \mathbf{N}$ such that $J_{j_0}, C_1, \ldots, C_m$ covers S, $\widehat{J_{j_0}} \subseteq C_0$ and $\mathrm{fdiam}(j_0) < r$. Now we start with the finite sequence $J_{j_0}, C_1, \ldots, C_m$ and get $j_1 \in \mathbf{N}$ such that $J_{j_0}, J_{j_1}, C_2, \ldots, C_m$ covers S, $\widehat{J_{j_1}} \subseteq C_1$ and $\mathrm{fdiam}(j_1) < r$. In finitely many steps we get the desired numbers. $\square$

Lemma 29. *Let (X, d) be a metric space and let S be a subset of X which is, as a subspace of (X, d), a (circularly) chainable continuum. Then for each $\varepsilon > 0$ there exists an open $\varepsilon-$(circular) quasi-chain $C_0, \ldots, C_m$ in (X, d) which covers S.*

Proof. Suppose S is chainable and let $\varepsilon > 0$. Let $D_0, \ldots, D_m$ be an $\frac{\varepsilon}{3}-$chain in S which covers S. By Lemma 28(i) there exist a finite sequence $A_0, \ldots, A_m$ of compact subsets of S which covers S and such that $A_i \subseteq D_i$, $i \in \{0, \ldots, m\}$. Let

$$r = \min\{d(A_i, A_j) \mid |i-j| > 1,\ i, j \in \{0, \ldots, m\}\}.$$

Then $r > 0$ since $A_i \cap A_j \subseteq D_i \cap D_j = \emptyset$ for $|i - j| > 1$. For $i \in \{0, \ldots, m\}$ let

$$C_i = \left\{x \in X \mid d(x, A_i) < \min\left\{\frac{r}{2}, \frac{\varepsilon}{3}\right\}\right\}.$$

Then C_i is an open set, $A_i \subseteq C_i$ and $C_i \cap C_j = \emptyset$ for all $i, j \in \{0, \ldots, m\}$ such that $|i - j| > 1$. Since $\mathrm{diam}(A_i) < \frac{\varepsilon}{3}$, we have $\mathrm{diam}(C_i) < \varepsilon$, $\forall i \in \{0, \ldots, m\}$. Therefore $C_0, \ldots, C_m$ is an open $\varepsilon-$quasi-chain in X which covers S. In the same way we get that there exists an open $\varepsilon-$ circular quasi-chain in X which covers S if S is circularly chainable. $\square$

Lemma 30. *Let $C_0, .., C_m$ be an open quasi-chain in (X, d) which covers S, where S is a connected subset of (X, d). Let $v = \min\{i \mid C_i \cap S \neq \emptyset\}$, $w = \max\{i \mid C_i \cap S \neq \emptyset\}$. Then $C_v, \ldots, C_w$ is a chain which covers S and $C_i \cap S \neq \emptyset$, $\forall i \in \{v, \ldots, w\}$.*

Proof. If $C_i \cap S = \emptyset$ for some i such that $v < i < w$, then $C_0, \ldots, C_{i-1}$ and $C_{i+1}, \ldots, C_m$ are disjoint open sets which cover S and each of them intersects S. This is impossible since S is connected. Therefore $C_i \cap S \neq \emptyset$ for all $i \in \{v, \ldots, w\}$. If i is such that $v \leq i < w$, then $C_i \cap C_{i+1} \neq \emptyset$ (otherwise $C_v \cup \ldots \cup C_i$ and $C_{i+1} \cup \ldots \cup C_w$ are disjoint open sets which cover S and each of them intersects S). Therefore $C_v, \ldots, C_w$ is a chain. $\square$

For $l \in \mathbf{N}$ let $\mathcal{H}_l$ be the finite sequence of sets $J_{(l)_0}, \ldots, J_{(l)_{\overline{l}}}$ and let $\widehat{\mathcal{H}}_l$ be the finite sequence of sets $\widehat{J}_{(l)_0}, \ldots, \widehat{J}_{(l)_{\overline{l}}}$

Let the function fmesh : $\mathbf{N} \to \mathbf{R}$ be defined by

$$\text{fmesh}(l) = \max_{0 \le j \le \bar{l}} \text{fdiam}((l)_j),$$

$l \in \mathbf{N}$. It is immediate from Proposition 13 and Proposition 2 that fmesh is a recursive function.

Let $\mathcal{C} = (C_0, \dots, C_m)$ and $\mathcal{D} = (D_0, \dots, D_n)$ be finite sequences of subsets of X. We say that $\mathcal{D}$ **directly refines** $\mathcal{C}$ if $n = m$ and $D_i \subseteq C_i$ for each $i \in \{0, \dots, m\}$.

Lemma 31. *Let (X, d, α) be a computable metric space, S compact subset of (X, d), $r \in \mathbf{R}$ and $\mathcal{C}$ an open $r-$(circular) quasi-chain in (X, d) which covers S. Then there exist $l \in \mathbf{N}$ such that $\mathcal{H}_l$ covers S, $\widehat{\mathcal{H}}_l$ directly refines $\mathcal{C}$ and* $\text{fmesh}(l) < r$.

Proof. This is immediate from Lemma 28(ii) and the fact that for all $j_0, \dots, j_m \in \mathbf{N}$ there exists $l \in \mathbf{N}$ such that $((l)_0, \dots, (l)_{\bar{l}}) = (j_0, \dots, j_m)$. □

Proposition 32. *Let (X, d, α) be a computable metric space which has the effective covering property and compact closed balls. The sets $\Omega = \{l \in \mathbf{N} \mid \widehat{\mathcal{H}}_l$ is a circular quasi-chain$\}$ and $\Omega' = \{l \in \mathbf{N} \mid \widehat{\mathcal{H}}_l$ is a quasi-chain$\}$ are recursively enumerable.*

Proof. Let $\Phi : \mathbf{N} \to \mathcal{P}(\mathbf{N}^2)$ be defined by $\Phi(l) = \{(i, j) \in \mathbf{N}^2 \mid i + 1 < j < \bar{l}$ or $1 < i + 1 < j \le \bar{l}\}$. Clearly, Φ is r.r.b. Let $\Psi : \mathbf{N}^3 \to \mathcal{P}(\mathbf{N}^2)$ be defined by $\Psi(i, j, l) = \{((l)_i, (l)_j)\}$. By Proposition 3(i) Ψ is r.r.b. Let $\Lambda : \mathbf{N} \to \mathcal{P}(\mathbf{N}^2)$ be defined by

$$\Lambda(l) = \{((l)_i, (l)_j) \mid i + 1 < j < \bar{l} \text{ or } 1 < i + 1 < j \le \bar{l}\}.$$

Since $\Lambda(l) = \cup_{(i,j) \in \Phi(l)} \Psi(i, j, l)$, Λ is r.r.b. (Proposition 4). Let S be the set of Proposition 16. Then $\Omega = \{l \in \mathbf{N} \mid \Lambda(l) \subseteq S\}$. By Lemma 5 Ω is r.e. We have $\Omega' = \Omega \cap (\{l \in \mathbf{N} \mid \widehat{J}_{(l)_0} \cap \widehat{J}_{(l)_{\bar{l}}} = \emptyset\} \cup \{l \in \mathbf{N} \mid \bar{l} \le 1\})$. Therefore Ω' is r.e. □

Lemma 33. *Let (X, d, α) be a computable metric space. There exists a recursive function $\zeta : \mathbf{N} \to \mathbf{N}$ such that $J_{\zeta(l)} = \bigcup \mathcal{H}_l$, $\forall l \in \mathbf{N}$.*

Proof. Let $\Phi : \mathbf{N} \to \mathcal{P}(\mathbf{N})$ be defined by $\Phi(l) = \cup_{0 \le i \le \bar{l}} [(l)_i]$. It follows from Proposition 4 that Φ is r.r.b. For each $l \in \mathbf{N}$ the set $\Phi(l)$ is nonempty and therefore there exists $j \in \mathbf{N}$ such that $\Phi(l) = [j]$. By Proposition 3(v) there exists a recursive function $\zeta : \mathbf{N} \to \mathbf{N}$ such that $\Phi(l) = [\zeta(l)]$. We have

$$J_{\zeta(l)} = \bigcup_{j \in [\zeta(l)]} I_j = \bigcup_{j \in \Phi(l)} I_j = \bigcup_{0 \le i \le \bar{l}} \left(\bigcup_{j \in [(l)_i]} I_j \right) = \bigcup_{0 \le i \le \bar{l}} J_{(l)_i}$$

and it is immediate from the definition of $\mathcal{H}_l$ that $J_{\zeta(l)}$ is its union. □

Let (X, d, α) be a computable metric space which has the effective covering property and compact closed balls and let S be a co-recursively enumerable set in (X, d, α) which is compact. Let $i_0 \in \mathbf{N}$ be such that $S \subseteq I_{i_0}$ and let $f : \mathbf{N} \to \mathbf{N}$ be a recursive function such that $X \setminus S = \bigcup_{n\in\mathbf{N}} J_{f(n)}$, $J_{f(n)} \subseteq J_{f(n+1)}$, $\forall n \in \mathbf{N}$. Let $j \in \mathbf{N}$. Then $S \subseteq J_j$ if and only if there exists $n \in \mathbf{N}$ such that $\widehat{I}_{i_0} \subseteq J_j \cup J_n$. It is not hard to conclude from this that the set $\{j \in \mathbf{N} \mid S \subseteq J_j\}$ is r.e. (see also Corollary 4.14 in [Brattka and Presser 2003]).

Proposition 34. *Let (X, d, α) be a computable metric space which has the effective covering property and compact closed balls. Let S be a co-recursively enumerable set in (X, d, α) which is compact. Then the set $\Omega = \{l \in \mathbf{N} \mid \mathcal{H}_l$ covers $S\}$ is r.e.*

Proof. Let $\zeta : \mathbf{N} \to \mathbf{N}$ be the function of Lemma 33. Then $\mathcal{H}_l$ covers S if and only if $S \subseteq J_{\zeta(l)}$, $\forall l \in \mathbf{N}$. Therefore $\Omega = \{l \in \mathbf{N} \mid \zeta(l) \in \Omega'\}$ where $\Omega' = \{j \in \mathbf{N} \mid S \subseteq J_j\}$. It follows from Proposition 1(ii) that Ω is r.e. □

Let (X, d, α) and S be as in Proposition 34. It is not hard to conclude from Proposition 34 that the set $\{i \in \mathbf{N} \mid S \subseteq I_i\}$ is r.e. Now, if S is a one-point set, then $S \cap I_i \neq \emptyset \Leftrightarrow S \subseteq I_i$, $\forall i \in \mathbf{N}$, which implies that S is an r.e. set. Hence each one point co-r.e. set in (X, d, α) is recursive.

5.1 Circularly chainable continua which are not chainable

Theorem 35. *Let (X, d, α) be a computable metric space which has the effective covering property and compact closed balls. Let S be a subset of X which, as a subspace of (X, d), is circularly chainable, but not chainable continuum. Then S is recursive if it is co-recursively enumerable.*

Proof. Suppose S is co-r.e. Let $k_0 \in \mathbf{N}$ be such that there exists no $2^{-k_0}-$ chain in S which covers S. If $C_0, \dots, C_m$ is an open $2^{-k_0}-$quasi-chain in X which covers S, then $C_0 \cap S, \dots, C_m \cap S$ is an open $2^{-k_0}-$quasi-chain in S which covers S and Lemma 30 implies that $C_i \cap S, \dots, C_j \cap S$ is a 2^{-k_0} chain in S which covers S for some $i, j \in \{0, \dots, m\}$, $i \leq j$, which is impossible. Hence there exists no open $2^{-k_0}-$quasi-chain in X which covers S.

Let $A = \{(k, l) \in \mathbf{N}^2 \mid \mathcal{H}_l$ covers S, $\widehat{\mathcal{H}}_l$ is a circular quasi-chain, $\text{fmesh}(l) < 2^{-(k+k_0)}\}$. By Proposition 34, Proposition 32 and Proposition 2(iv) the set A is r.e.

Let $k \in \mathbf{N}$. By Lemma 29 there exists an open circular $2^{-(k+k_0)}-$quasi-chain $\mathcal{C} = (C_0, \dots, C_m)$ in X which covers S. By Lemma 31 there exists $l \in \mathbf{N}$ such that $\mathcal{H}_l$ covers S, $\widehat{\mathcal{H}}_l$ refines $\mathcal{C}$ and $\text{fmesh}(l) < 2^{-(k+k_0)}$. Since $\widehat{\mathcal{H}}_l$ refines $\mathcal{C}$, $\widehat{\mathcal{H}}_l$ is a quasi-chain and we conclude that $(l, k) \in A$. Hence for each $k \in \mathbf{N}$ there exists

$l \in \mathbf{N}$ such that $(k,l) \in A$. By Proposition 1(i) there exists a recursive function $\varphi : \mathbf{N} \to \mathbf{N}$ such that $(k, \varphi(k)) \in A$, $\forall k \in \mathbf{N}$.

Let $k \in \mathbf{N}$ and let $\mathcal{H}_{\varphi(k)} = (C_0, \dots, C_m)$. Then $C_0, \dots, C_m$ is an open quasi-chain in X which covers S. Since $\mathrm{diam}(J_j) \leq \mathrm{fdiam}(j)$, $\forall j \in \mathbf{N}$ and $\mathrm{fmesh}(\varphi(k)) < 2^{-(k+k_0)}$, we conclude that $\mathrm{diam}(C_i) < 2^{-(k+k_0)}$, $\forall i \in \{0, \dots, m\}$.

Suppose that there exists $i \in \{0, \dots, m\}$ such that $C_i \cap S = \emptyset$. Then $C_{i+1}, \dots, C_m, C_0, \dots, C_{i-1}$ covers S. But $C_{i+1}, \dots, C_m, C_0, \dots, C_{i-1}$ is a quasi-chain (since $C_0, \dots, C_m$ is a circular quasi-chain). Hence we have an open $2^{-k_0}-$ quasi-chain in X which covers S which is impossible. Therefore $C_i \cap S \neq \emptyset$, $\forall i \in \{0, \dots, m\}$. We have $\{C_0, \dots, C_m\} = \{J_j \mid j \in [\varphi(k)]\}$ and by Lemma 14 S is a recursive set. $\square$

A continuum K is said to be **decomposable** if it is the union of two proper subcontinua. We say that K is **indecomposable** if it is not decomposable. For example $[0,1]$ is a decomposable continuum since $[0,1] = \left[0, \frac{1}{2}\right] \cup \left[\frac{1}{2}, 1\right]$ (or $[0,1] = \left[0, \frac{2}{3}\right] \cup \left[\frac{1}{3}, 1\right]$). On the other hand, it is much harder to prove that there exists an indecomposable continuum (apart from a point), an example of such a continuum can be found in [Nadler 1992] (Example 1.10).

We say that a continuum K is **2-indecomposable** if it is decomposable and if there exist no subcontinua M_1, M_2 and M_3 of K whose union is K and such that neither of M_1, M_2 and M_3 is contained in the union of other two [Burgess 1959].

There exist continua which are both chainable and circularly chainable. There is a result proved in [Burgess 1959] which says that if K is a chainable continuum, then K is circularly chainable if and only if it is indecomposable or 2-indecomposable. For example, if $K = [0,1]$ or $K = S^1$, then K is neither indecomposable nor 2-indecomposable. Therefore, since $[0,1]$ is chainable and S^1 is circularly chainable, $[0,1]$ is not circularly chainable and S^1 is not chainable.

If K and K' are homeomorphic metric spaces, then it is not hard to see that K is a (circularly) chainable continuum if and only if K' is a (circularly) chainable continuum.

In the following examples let X be an arbitrary computable metric space which has the effective covering property and compact closed balls.

Example 5. (i) If $S \subseteq X$ is homeomorphic to S^1, then S is circularly chainable, but not chainable. Therefore, by Theorem 35, S is recursive if it is co-r.e.

(ii) Let T be the subset of $\mathbf{R}^2$ defined by

$$T = \{(0,s) \mid s \in [-1,1]\} \cup \left\{ \left(t, \sin\left(\frac{1}{t}\right) \mid t \in \langle 0, 1] \right\} .$$

Let A be an arc whose endpoints are $(0,-1)$ and $(1, \sin(1))$ and which intersects T only in these points. Then any space homeomorphic to $T \cup A$ is called the

Warsaw circle. It is easy to conclude that $T \cup A$ is circularly chainable, but not chainable since it is clearly decomposable and not 2-indecomposable. Therefore each co-r.e. subset of X which is homeomorphic to $T \cup A$ must be recursive, i.e. each co-r.e. Warsaw circle in X is recursive.

Example 6. Let $f : S^1 \to S^1$ be defined by $f(z) = z^2$, where $z^2 = z \cdot z$ is multiplication in $\mathbf{C}$. Let A be set of all sequences (x_i) in S^1 and $B = \{(x_i) \in A \mid x_i = f(x_{i+1}),\ \forall i \in \mathbf{N}\}$. We have the product topology on A ($A = S^1 \times S^1 \times \ldots$) and B with the subspace topology. The space B is called the dyadic solenoid [Nadler 1992, Christenson and Voxman 1977]. As a consequence of Theorem 35 we have that each co-r.e. set in X which is homeomorphic to the dyadic solenoid must be recursive. Namely, by [Nadler 1992], the dyadic solenoid is a circularly chainable continuum which is not chainable and, interestingly, this is an indecomposable continuum.

Let (S_i) be a sequence of closed sets in a computable metric space (X, d, α). We say that (S_i) is a co-recursively enumerable sequence if there exists an r.e. set $A \subseteq \mathbf{N}^2$ such that for each $i \in \mathbf{N}$ the set $X \setminus S_i$ is the union of all I_j such that $(i, j) \in A$. We say that (S_i) is a recursively enumerable sequence if the set $\{(i, j) \mid S_i \cap I_j \neq \emptyset\}$ is r.e. A sequence (S_i) is said to be recursive if it is r.e. and co-r.e.

Suppose (X, d, α) is a computable metric space which has the effective covering property and compact closed balls. In view of Theorem 35 it makes sense to ask the following question: if (S_i) is a co-r.e. sequence of sets and each S_i a circularly chainable, but not chainable continuum, is then the sequence (S_i) recursive? The following example shows that this in general does not hold. We will use here the fact that a sequence of nonempty closed sets in $\mathbf{R}^n$ is recursive if and only if the sequence of distance functions $d_{S_i} : \mathbf{R}^n \to \mathbf{R}$, $d_{S_i}(x) = d(x, S_i)$, is computable and that (S_i) is co-r.e. if and only if there exists a computable sequence of functions $f_i : \mathbf{R}^n \to \mathbf{R}$ such that $S_i = f_i^{-1}(\{0\})$ [Weihrauch 2000].

Example 7. Let (λ_i) be a recursive sequence of real numbers such that $\lambda_i = 0$ cannot be decided effectively for $i \in \mathbf{N}$ [Pour-El and Richards 1989, p. 23]. We may assume that $0 \leq \lambda_i \leq \frac{1}{4}$, $\forall i \in \mathbf{N}$. For each $i \in \mathbf{N}$ let a_i, b_i, c_i, d_i be points in $\mathbf{R}^2$ defined by

$$a_i = (\lambda_i, 1),\ b_i = (\lambda_i, \lambda_i),\ c_i = (1 - \lambda_i, \lambda_i),\ d_i = (1 - \lambda_i, 1).$$

Let $A = (0, 1)$, $B = (0, 0)$, $C = (1, 0)$ and $D = (1, 1)$. For $i \in \mathbf{N}$ let T_i and T_i' be subsets of $\mathbf{R}^2$ defined as unions of segments:

$$T_i = \overline{AB} \cup \overline{BC} \cup \overline{CD},\ \ T_i' = \overline{Aa_i} \cup \overline{a_i b_i} \cup \overline{b_i c_i} \cup \overline{c_i d_i} \cup \overline{d_i D}.$$

The sequence of sets $(T_i \cup T_i')_{i \in \mathbf{N}}$ is recursive which follows from the fact that $(\overline{P_i Q_i})$ is a recursive sequence of sets for all recursive sequences (P_i), (Q_i) in $\mathbf{R}^2$.

Therefore there exists a computable sequence of functions (h_i), $h_i : \mathbf{R}^2 \to \mathbf{R}$ such that $h_i^{-1}(\{0\}) = T_i \cup T_i'$, $\forall i \in \mathbf{N}$. Let $g : \mathbf{R}^2 \to \mathbf{R}$ be a nonnegative computable function such that $g^{-1}(\{0\}) = \overline{AD}$. For $i \in \mathbf{N}$ let $f_i : \mathbf{R}^2 \to \mathbf{R}$ be defined by $f_i(x) = h_i(x) \cdot (g(x) + \lambda_i)$, $x \in \mathbf{R}^2$. Then (f_i) is a computable sequence of functions. Let $S_i = f_i^{-1}(\{0\})$, $i \in \mathbf{N}$. Let $i \in \mathbf{N}$. Then $S_i = T_i \cup T_i'$ if $\lambda_i > 0$, and $S_i = T_i \cup \overline{AD}$ if $\lambda_i = 0$. It follows that S_i is a topological circle. Let $E = (\frac{1}{2}, 1)$. Then $d(E, S_i) = \frac{1}{2} - \lambda_i$ if $\lambda_i > 0$ and $d(E, S_i) = 0$ if $\lambda_i = 0$, hence $d(E, S_i) \geq \frac{1}{4}$ if $\lambda_i > 0$, $d(E, S_i) = 0$ if $\lambda_i = 0$. This and the fact that $\lambda_i = 0$ cannot be decided effectively for $i \in \mathbf{N}$ imply that the sequence $(d(E, S_i))_{i \in \mathbf{N}}$ is not recursive. Conclusion: (S_i) is a co-r.e. sequence of topological circles, but (S_i) is not recursive.

5.2 Chainable decomposable continua

A continuum K is said to be **chainable from** a **to** b, $a, b \in X$, if for each $\varepsilon > 0$ there exists an $\varepsilon-$chain $C_0, \dots, C_m$ in K which covers K and such that $a \in C_0$, $b \in C_m$ [Christenson and Voxman 1977]. For example $[0, 1]$ is a continuum chainable from 0 to 1. If $f : K \to K'$ is a homeomorphism and $a, b \in K$, then it is not hard to see that K is chainable from a to b if and only if K' is chainable from $f(a)$ to $f(b)$. Therefore, if A is an arc and a and b its endpoints, then A is a continuum chainable from a to b.

Theorem 36. *Let (X, d, α) be a computable metric space which has the effective covering property and compact closed balls. Let $S \subseteq X$ be a co-r.e. set which, as a subspace, is a continuum chainable from a, b, where a and b are recursive points in X. Then S is recursive.*

Proof. It is easy to see that, since a is a recursive point, the set $\{i \in \mathbf{N} \mid a \in I_i\}$ is r.e. Since $a \in J_j \Leftrightarrow \exists i \in [j]\ a \in I_i$, we have that $\{j \in \mathbf{N} \mid a \in J_j\}$ is r.e. and also $\{j \in \mathbf{N} \mid b \in J_j\}$ is r.e.

Let $k \in \mathbf{N}$. Then there exists an $\frac{2^{-k}}{3}-$chain $D_0, \dots, D_m$ in S which covers S such that $a \in D_0$, $b \in D_m$. We may assume that $a \notin D_1 \cup \dots \cup D_m$, $b \notin D_0 \cup \dots \cup D_{m-1}$, otherwise we may start with an $\frac{2^{-k}}{6}-$chain $D_0, \dots, D_m$ and then take the $\frac{2^{-k}}{3}-$chain $D_0 \cup D_1, D_2, \dots, D_{m-2}, D_{m-1} \cup D_m$ (we may assume $m \geq 3$ if $a \neq b$; if $a = b$, then $S = \{a\}$ and S is recursive). As in the proof of Lemma 29 we get an $2^{-k}-$quasi chain in X which covers S such that $a \in C_0$, $b \in C_m$. Again, we may assume that $a \notin C_1 \cup \dots \cup C_m$, $b \notin C_0 \cup \dots, C_{m-1}$. By Lemma 31 there exist $l \in \mathbf{N}$ such that $\mathcal{H}_l$ covers S, $\widehat{\mathcal{H}}_l$ directly refines $\mathcal{C}$ and $\text{fmesh}(l) < 2^{-k}$. It follows that $\widehat{\mathcal{H}}_l$ is a quasi-chain and $a \in J_{(l)_0}$, $b \in J_{(l)_{\overline{l}}}$. As in the proof of Theorem 35 we conclude that there exists a recursive function $\varphi : \mathbf{N} \to \mathbf{N}$ such that $\mathcal{H}_{\varphi(k)}$ covers S, $\widehat{\mathcal{H}}_{\varphi(k)}$ is a quasi-chain, $a \in J_{(\varphi(k))_0}$, $b \in J_{(\varphi(k))_{\overline{\varphi(k)}}}$ and $\text{fmesh}(l) < 2^{-k}$, $\forall k \in \mathbf{N}$. By Lemma 30 for each $k \in \mathbf{N}$ each link of the chain $\mathcal{H}_{\varphi(k)}$ intersects S and therefore, by Lemma 14, S is a recursive set. □

Example 8. Let T be the space defined in Example 5. This space is known under the name Topologist's sine curve. Let $a = (0, -1)$, $b = (0, 1)$ and $c = (1, \sin(1))$. Then T is chainable from a to c and it is also chainable from b to c. Suppose X is a computable metric space which has the effective covering property and compact closed balls. Let $f : T \to X$ is an embedding such that $f(T)$ is a co-r.e. set and such that $f(c)$ is a recursive point and $f(a)$ or $f(b)$ is a recursive point. Then, by Theorem 36, $f(T)$ is a recursive set.

In contrast to the fact that each co-r.e. set $S \subseteq X$ homeomorphic to S^1 must be recursive (under some assumptions on X), there are co-r.e. arcs in X which are not recursive. In fact there exists a co-r.e. segment in $\mathbf{R}$ which is not recursive [Miller 2002]. Hence the implication S co-r.e. $\Rightarrow$ S recursive fails to be true even for very simple chainable continua. Although co-r.e. chainable decomposable continuum S need not be recursive in general, recursive points of S must be dense in S (under some assumptions on X), furthermore we are going to prove that for each $\varepsilon > 0$ we can "$\varepsilon-$approximate" S with a subcontinuum which is recursive.

Let $\mathcal{C} = (C_0, \dots, C_m)$ and $\mathcal{D} = (D_0, \dots, D_n)$ be finite sequences of subsets of X. We say that $\mathcal{D}$ **refines** $\mathcal{C}$ if for each $i \in \{0, \dots, n\}$ there exists $j \in \{0, \dots, m\}$ such that $D_i \subseteq C_j$. If $\mathcal{D}$ refines $\mathcal{C}$ and $D_0 \subseteq C_0$, $D_n \subseteq C_m$, then we say that $\mathcal{D}$ **strongly refines** $\mathcal{C}$.

Lemma 37. *Let (X, d, α) be a computable metric space which has the effective covering property and compact closed balls. Let $\Omega = \{(l, v, w, l', v', w') \in \mathbf{N}^6 \mid v \leq w \leq \bar{l}, v' \leq w' \leq \overline{l'}$ and $(\widehat{J}_{(l')_{v'}}, \dots, \widehat{J}_{(l')_{w'}})$ strongly refines $(J_{(l)_v}, \dots, J_{(l)_w})\}$. Then Ω is a recursively enumerable set.*

Proof. Let $S = \{(k, l, v, w) \in \mathbf{N}^4 \mid \exists j$ such that $v \leq j \leq w$ and $\widehat{J}_k \subseteq J_{(l)_j}\}$. By Proposition 15 and Proposition 1(i) S is r.e. Let $\Phi : \mathbf{N}^6 \to \mathcal{P}(\mathbf{N}^4)$, $\Phi(l, v, w, l', v', w') = \{((l')_i, l, v, w) \mid v' \leq i \leq w'\}$. It follows from Proposition 4 that Φ is an r.r.b. function. Let

$$\Omega_1 = \{(l, v, w, l', v', w') \mid \Phi(l, v, w, l', v', w') \subseteq S\},$$

$$\Omega_2 = \{(l, v, w, l', v', w') \mid v \leq w \leq \bar{l},\ v' \leq w' \leq \overline{l'}\},$$

$$\Omega_3 = \{(l, v, w, l', v', w') \mid \widehat{J}_{(l')_{v'}} \subseteq J_{(l)_v},\ \widehat{J}_{(l')_{w'}} \subseteq J_{(l)_w}\}.$$

By Lemma 5 Ω_1 is r.e. It follows from Proposition 15 and Proposition 1(ii) that Ω_3 is the intersection of two r.e. sets, therefore Ω_3 is r.e. Clearly Ω_2 is recursive. That Ω is recursively enumerable follows now from $\Omega = \Omega_1 \cap \Omega_2 \cap \Omega_3$. □

Lemma 38. *Let T be a compact set in (X,d) and $C_0,\dots,C_m$ a finite sequence of open sets which covers T. Then there exists $\varepsilon_0>0$ with the following property: if $D_0,\dots,D_n$ is a finite sequence of ε_0-bounded sets such that $D_i\cap T\neq\emptyset$, $\forall i\in\{0,\dots,n\}$, then $(D_0,\dots,D_n)$ refines $(C_0,\dots,C_m)$.*

Proof. Let $r=d(T,X\setminus(C_0\cup\ldots\cup C_m))$. Then $r>0$ since T is compact. Let $d_T:X\to\mathbf{R}$, $d_T(x)=d(x,T)$, $x\in X$ and $T'=d_T^{-1}([0,\frac{r}{2}])$. Then T' is a closed subset of (X,d), and since T is bounded, T' is bounded. Since closed balls in (X,d) are compact, T' as a closed and bounded set must be compact. It follows from the definition of T' that $\{C_0,\dots,C_m\}$ is an open cover of T'. Let $\lambda>0$ be its Lebesgue number, hence $S\subseteq T'$, $\mathrm{diam}(S)<\lambda$ implies $S\subseteq C_i$ for some i. Let $\varepsilon_0=\min\{\frac{r}{2},\lambda\}$.

Suppose that $D_0,\dots,D_n$ is a finite sequence of ε_0-bounded sets such that $D_i\cap T\neq\emptyset$, $\forall i\in\{0,\dots,n\}$. Let $i\in\{0,\dots,n\}$. Since $D_i\cap T\neq\emptyset$, we have $d(x,T)<\mathrm{diam}(D_i)$, $\forall x\in D_i$, which together with $\mathrm{diam}(D_i)<\varepsilon_0\leq\frac{r}{2}$ implies $D_i\subseteq T'$. Now $\mathrm{diam}(D_i)<\lambda$ implies that $D_i\subseteq C_j$ for some $j\in\{0,\dots,m\}$. □

Lemma 39. *Let $D_0,\dots,D_n$ be a chain which refines a chain $C_0,\dots,C_m$ and let $i_0,j_0\in\{0,\dots m\}$ be such that $i_0\leq j_0$ and $D_0\subseteq C_{i_0}$, $D_n\subseteq C_{j_0}$. Then for each $i,j\in\{i_0,\dots,j_0\}$, $i\leq j$ there exist $i',j'\in\{0,\dots,n\}$ such that $i'\leq j'$ and $D_{i'},\dots,D_{j'}$ strongly refines $C_i,\dots,C_j$.*

Proof. Let

$$v=\max\{k\in\{0,\dots,n\}\mid D_k\subseteq C_l \text{ for some } l\in\{0,\dots,i\}\}.$$

We claim that $D_v\subseteq C_i$. Suppose opposite. Then $D_v\subseteq C_l$, $l<i$. It follows $v<n$ and $D_{v+1}\subseteq C_{l'}$, $l'>i$. However, this implies

$$\emptyset\neq D_v\cap D_{v+1}\subseteq C_l\cap C_{l'}=\emptyset,$$

a contradiction. Hence $D_v\subseteq C_i$. If $v=n$, then we are finished since $D_v\subseteq C_i\cap C_m$ implies $i=m$ or $i=m-1$. Therefore, assume $v<n$. We may also assume $i<j$. Let

$$w=\min\{k\in\{v+1,\dots,n\}\mid D_k\subseteq C_l \text{ for some } l\in\{j,\dots,m\}.$$

Then $w\geq v+1$ and $D_w\subseteq C_l$ for some $l\geq j$. Inequality $l>j$ easily yields to contradiction, therefore $D_w\subseteq C_j$.

Finally, let k be such that $v<k<w$. Then it is clear from the definitions of v and w that $D_k\subseteq C_l$, where l is such that $i<l<j$. □

Lemma 40. *Let (X,d,α) be a computable metric space which has the effective covering property and compact closed balls. Let A and B be subsets of X of the form $A=\widehat{J}_a$, $B=\widehat{J}_b$, $a,b\in\mathbf{N}$. Then the sets $V=\{(l,v)\mid A\cap\widehat{J}_{(l)_i}=\emptyset$, for all i such that $v\leq i\leq\overline{l}\}$ and $W=\{(l,w)\mid B\cap\widehat{J}_{(l)_i}=\emptyset$, for all i such that $0\leq i\leq w\}$ are recursively enumerable.*

Proof. Let $\Phi : \mathbf{N}^2 \to \mathcal{P}(\mathbf{N})$ be defined by $\Phi(l, v) = \{(l)_i \mid v \leq i \leq \bar{l}\}$. Then Φ is r.r.b. by Proposition 4. The set $\Omega = \{i \in \mathbf{N} \mid \widehat{J}_a \cap \widehat{J}_i = \emptyset\}$ is r.e. by Proposition 16. We have $V = \{(l, v) \in \mathbf{N}^2 \mid \Phi(l, v) \subseteq \Omega\}$, therefore V is r.e. by Lemma 5. It follows in the same way that W is r.e. □

Lemma 41. *Let (X, d) be a metric space which has compact closed balls. Let $\mathcal{C}^k = (C_0^k, \dots, C_{m_k}^k)$, $k \in \mathbf{N}$ be a sequence of chains such that $\overline{C_0^{k+1}}, \dots, \overline{C_{m_{k+1}}^{k+1}}$ strongly refines $C_0^k, \dots, C_{m_k}^k$ and* $\mathrm{mesh}(\mathcal{C}^k) < 2^{-k}$, $\forall k \in \mathbf{N}$. *Let*

$$S = \bigcap_{k \in \mathbf{N}} \left(\overline{C_0^{k+1}} \cup \ldots \cup \overline{C_{m_{k+1}}^{k+1}} \right).$$

Then S is a continuum chainable from a to b, where $a \in \bigcap_{k \in \mathbf{N}} C_0^k$, $b \in \bigcap_{k \in \mathbf{N}} C_{m_k}^k$.

Proof. Since (X, d) has compact closed balls, $T \subseteq X$ is compact if and only if it is closed and bounded. We conclude that S is compact and $\cap_{k \in \mathbf{N}} \overline{C_0^k} = \{a\}$, $\cap_{k \in \mathbf{N}} \overline{C_{m_k}^k} = \{b\}$, where $a, b \in X$. It follows $a, b \in S$, $\cap_{k \in \mathbf{N}} C_0^k = \{a\}$, $\cap_{k \in \mathbf{N}} C_{m_k}^k = \{b\}$ and $S \subseteq \bigcup \mathcal{C}^k$, $\forall k \in \mathbf{N}$. It remains to prove that S is connected.

Suppose that S is not connected. Then there exist closed, nonempty and disjoint subsets K_1 and K_2 of S whose union is S. It follows that K_1 and K_2 are compact. Let $\lambda = d(K_1, K_2)$. Then $\lambda > 0$ and there exists $k \in \mathbf{N}$ such that $2^{-k} < \frac{\lambda}{2}$. We may assume $a \in K_1$. Let $i_0 = \min\{i \mid C_i^k \cap K_2 \neq \emptyset\}$. Then $i_0 \geq 1$ since $i_0 = 0$ implies $C_0 \cap K_2 \neq \emptyset$ which is in contradiction with $a \in K_1 \cap C_0^K$, $\mathrm{diam}(C_0^k) < \lambda$, $d(K_1, K_2) = \lambda$. Now $C_{i_0-1}^k \cap K_2 = \emptyset$. But we also have $C_{i_0-1}^k \cap K_1 = \emptyset$, namely $C_{i_0-1}^k \cap K_1 \neq \emptyset$, $C_{i_0-1}^k \cap C_{i_0}^k \neq \emptyset$, $C_{i_0}^k \cap K_2 \neq \emptyset$ and $\mathrm{mesh}(\mathcal{C}^k) < \frac{\lambda}{2}$ is in contradiction with $d(K_1, K_2) = \lambda$. Therefore $C_{i_0-1}^k \cap S = \emptyset$. Using Lemma 39 we obtain inductively a sequence $j_1, j_2, \dots$ such that $C_{i_0-1}^k \supseteq C_{j_1}^{k+1} \supseteq C_{j_2}^{k+2} \supseteq \dots$. Then $\bigcap_{p \geq 1} \overline{C_{j_p}^{k+p}}$ is a nonempty subset of S and it is also subset of $C_{i_0-1}^k$. This is a contradiction with $C_{i_0-1}^k \cap S = \emptyset$. □

Theorem 42. *Let (X, d, α) be a computable metric space which has the effective covering property and compact closed balls. Let $S \subseteq X$ be a chainable continuum, $a, b \in S$ and K_1, K_2 subcontinua of S such that $S = K_1 \cup K_2$, $a \in K_1 \setminus K_2$, $b \in K_2 \setminus K_1$. Suppose S is a co-r.e. set in (X, d, α). Then for each $\varepsilon > 0$ there exist recursive points $a', b' \in S$ such that $d(a, a') < \varepsilon$, $d(b, b') < \varepsilon$ and a subcontinuum K of S which is recursive and chainable from a' to b'.*

Proof. Let

$$r = \min \left\{ d(a, K_2), d(b, K_1), \frac{\varepsilon}{3} \right\}.$$

Choose $\widetilde{a}, \widetilde{b} \in \mathbf{N}$ so that for the sets $A = \widehat{J}_{\widetilde{a}}$, $B = \widehat{J}_{\widetilde{b}}$ the following holds: $a \in A$, $b \in B$, $\mathrm{diam}(A) < \frac{r}{4}$, $\mathrm{diam}(B) < \frac{r}{4}$.

If $\mathcal{C} = (C_0, \ldots, C_m)$ is a finite sequence of sets which covers S, then we define numbers $v_\mathcal{C}, w_\mathcal{C}$ by

$$v_\mathcal{C} = \max\{i \in \{0, \ldots m\} \mid C_i \cap A \neq \emptyset\},\ w_\mathcal{C} = \min\{j \in \{0, \ldots m\} \mid C_j \cap B \neq \emptyset\}.$$

Claim 1 If $\mathcal{C} = (C_0, \ldots, C_m)$ is a quasi-chain which covers S and $v_\mathcal{C}+1 < w_\mathcal{C}$, then $C_{v_\mathcal{C}}, \ldots, C_{w_\mathcal{C}}$ is a chain and each link of this chain intersects S.

Indeed, it follows from the definition of $v_\mathcal{C}$ and $w_\mathcal{C}$ that $a \in C_0 \cup \ldots \cup C_{v_\mathcal{C}}$, $b \in C_{w_\mathcal{C}} \cup \ldots \cup C_m$, therefore $C_i \cap S \neq \emptyset$, $C_j \cap S \neq \emptyset$, for some i, j such that $i \leq v_\mathcal{C}$, $w_\mathcal{C} \leq j$. Now Claim 1 follows from Lemma 30.

Claim 2 For each $\delta > 0$ there exists a $\delta-$chain $\mathcal{C}$ which covers S such that $v_\mathcal{C} + 1 < w_\mathcal{C}$.

Let $\delta > 0$. Let $\mathcal{C} = (C_0, \ldots, C_m)$ be a $\min\{\delta, \frac{r}{4}\}-$chain which covers S such that $C_i \cap S \neq \emptyset$, $\forall i \in \{0, \ldots, m\}$ (Lemma 29, Lemma 30). Suppose that there exist $i, j, k \in \{0, \ldots, m\}$ such that $i < j < k$ and

$$C_i \cap A \neq \emptyset,\ C_j \cap B \neq \emptyset,\ C_k \cap A \neq \emptyset. \tag{9}$$

It follows from $C_i \cap A \neq \emptyset$ that $d(a, x) < r$, $\forall x \in C_i$. Therefore $C_i \cap K_2 = \emptyset$. Similarly $C_j \cap K_1 = \emptyset$, $C_2 \cap K_2 = \emptyset$. It follows $C_i \cap K_1 \neq \emptyset$, $C_k \cap K_1 \neq \emptyset$ and consequently $C_0 \cup \ldots \cup C_{j-1}$ and $C_{j+1} \cup \ldots \cup C_m$ are open disjoint sets which cover K_1 and each of these sets intersects K_1. This contradicts the fact that K_1 is connected. Hence there are no $i < j < k$ such that (9) holds. Similarly there are no $i < j < k$ such that $C_i \cap B \neq \emptyset$, $C_j \cap A \neq \emptyset$, $C_k \cap B \neq \emptyset$. Notice that there are no $i, j \in \{0, \ldots, m\}$ such that $|i - j| \leq 1$, $C_i \cap A \neq \emptyset$, $C_j \cap B \neq \emptyset$. We have the following conclusion:

$$\max\{i \mid C_i \cap A \neq \emptyset\} + 1 < \min\{j \mid C_j \cap B \neq \emptyset\} \text{ (i.e. } v_\mathcal{C} + 1 < w_\mathcal{C})$$

or

$$\max\{i \mid C_i \cap B \neq \emptyset\} + 1 < \min\{j \mid C_j \cap B \neq \emptyset\}.$$

If the second inequality holds, we take the chain $\mathcal{C}' = (C_m, \ldots, C_0)$ and then we have $v_{\mathcal{C}'} + 1 < w_{\mathcal{C}'}$. This completes the proof of Claim 2.

Claim 3 Let $\mathcal{C} = (C_0, \ldots, C_m)$ be an open quasi-chain which covers S and v, w numbers such that $v_\mathcal{C} < v \leq w < w_\mathcal{C}$. Then for each $r > 0$ there exists an open $r-$quasi-chain $\mathcal{D} = (D_0, \ldots, D_n)$ which covers S and $v', w' \in \{0, \ldots, n\}$ such that $v_\mathcal{D} < v' \leq w' < w_\mathcal{D}$ and such that $D_{v'}, \ldots, D_{w'}$ strongly refines $C_v, \ldots, C_w$.

By Lemma 29, Lemma 30, Lemma 38 and Claim 2 there exists an open $r-$chain $\mathcal{D} = (D_0, \ldots, D_n)$ which refines $\mathcal{C}$, covers S and such that $v_\mathcal{D}+1 < w_\mathcal{D}$, $D_k \cap S \neq \emptyset$, $k \in \{0, \ldots, n\}$. Let $i, j \in \{0, \ldots, m\}$ be such that $D_{v_\mathcal{D}} \subseteq C_i$, $D_{w_\mathcal{D}} \subseteq C_j$. It follows from the definition of $v_\mathcal{C}$ and $w_\mathcal{C}$ that $i \leq v_\mathcal{C}$, $w_\mathcal{C} \leq j$.

Therefore $i < v \leq w < j$. Since $D_{v_\mathcal{D}} \cap S \neq \emptyset$ (Claim 1), $C_i \cap S \neq \emptyset$. Similarly $C_j \cap S \neq \emptyset$. Let $i_0 = \min\{i \mid C_i \cap S \neq \emptyset\}$, $j_0 = \max\{i \mid C_i \cap S \neq \emptyset\}$. By Lemma 30 $C_{i_0}, \dots, C_{j_0}$ is a chain and since $D_k \cap S \neq \emptyset$, $\forall k \in \{0, \dots, n\}$, $\mathcal{D}$ refines this chain. In particular $D_{v_\mathcal{D}}, \dots, D_{w_\mathcal{D}}$ refines $C_{i_0}, \dots, C_{j_0}$ and since $i_0 \leq i < v \leq w < j \leq j_0$, by Lemma 39 there exist v', w' such that $v_\mathcal{D} \leq v' \leq w' \leq w_\mathcal{D}$ and such that $D_{v'}, \dots, D_{w'}$ strongly refines $C_v, \dots, C_w$. Notice that $D_{v'} \subseteq C_v$ and $v_\mathcal{D} < v$ imply $D_{v'} \cap A = \emptyset$, therefore $v_\mathcal{D} < v'$. Similarly $w' < w_\mathcal{D}$. This completes the proof of Claim 3.

Claim 4 Let $\mathcal{C} = (C_0, \dots, C_m)$ be an open quasi-chain which covers S and v, w numbers such that $v_\mathcal{C} < v \leq w < w_\mathcal{C}$. Then for each $r > 0$ there exists $l \in \mathbf{N}$ and $v', w' \in \{0, \dots, \overline{l}\}$ such that $\mathcal{H}_l$ covers S, $\widehat{\mathcal{H}}_l$ is a quasi-chain, $v_{\widehat{\mathcal{H}}_l} < v' \leq w' < w_{\widehat{\mathcal{H}}_l}$, $\widehat{J}_{(l)_{v'}}, \dots, \widehat{J}_{(l)_{w'}}$ strongly refines $C_v, \dots, C_w$ and $\operatorname{fmesh}(l) < r$.

There exist $\mathcal{D} = (D_0, \dots, D_n)$ and $v', w' \in \{0, \dots, n\}$ as in Claim 3. By Lemma 31 there exists $l \in \mathbf{N}$ such that $\mathcal{H}_l$ covers S, $\widehat{\mathcal{H}}_l$ directly refines $\mathcal{D}$ and $\operatorname{fmesh}(l) < r$. The fact that $\widehat{\mathcal{H}}_l$ directly refines $\mathcal{D}$ clearly implies that $v_{\widehat{\mathcal{H}}_l} \leq v_\mathcal{D}$, $w_\mathcal{D} \leq w_{\widehat{\mathcal{H}}_l}$ (hence $v_{\widehat{\mathcal{H}}_l} < v' \leq w' < w_{\widehat{\mathcal{H}}_l}$), that $\widehat{\mathcal{H}}_l$ is a quasi-chain and that $\widehat{J}_{(l)_{v'}}, \dots, \widehat{J}_{(l)_{w'}}$ strongly refines $C_v, \dots, C_w$. Hence Claim 4 holds.

Let $\Delta = \{(l, v, w) \in \mathbf{N}^3 \mid \mathcal{H}_l$ covers S, $\widehat{\mathcal{H}}_l$ is a quasi-chain and $v_{\widehat{\mathcal{H}}_l} < v \leq w < w_{\widehat{\mathcal{H}}_l}\}$.

Claim 5 The set Δ is recursively enumerable.

Let V and W be the sets associated to A and B as in Lemma 40. Let $l, v, w \in \mathbf{N}$. Then $\mathcal{H}_l$ covers S and $v_{\widehat{\mathcal{H}}_l} < v \leq w < w_{\widehat{\mathcal{H}}_l}$ if and only if $\mathcal{H}_l$ covers S, $(l, v) \in V$, $(l, w) \in W$ and $v \leq w$. Claim 5 now follows from Proposition 34, Proposition 32 and Lemma 40.

Let Γ be the set of all $(l, v, w, l', v', w') \in \mathbf{N}^6$ such that $v \leq w \leq \overline{l}$, $v' \leq w' \leq \overline{l'}$, $\widehat{J}_{(l')_{v'}}, \dots, \widehat{J}_{(l')_{w'}}$ strongly refines $J_{(l)_v}, \dots, J_{(l)_w}$ and $\operatorname{fmesh}(l') < \frac{1}{2} \operatorname{fmesh}(l)$. It follows from Lemma 37 and Proposition 2(iv) that Γ is recursively enumerable.

Claim 6 For each $(l, v, w) \in \Delta$ there exists $(l', v', w') \in \Delta$ such that $(l, v, w, l', v', w') \in \Gamma$.

Let $(l, v, w) \in \Delta$. Obviously $\mathcal{H}_l$ directly refines $\widehat{\mathcal{H}}_l$, therefore $\mathcal{H}_l$ is an open quasi-chain and $v_{\mathcal{H}_l} < v \leq w < w_{\mathcal{H}_l}$. Now we apply Claim 4 to $\mathcal{H}_l$, v, w and get l', v', w' such that $(l', v', w') \in \Delta$, $(l, v, w, l', v', w') \in \Gamma$.

It follows from Claim 2 and Claim 4 that there exists $p \in \mathbf{N}$ such that $\mathcal{H}_p$ covers S, $\widehat{\mathcal{H}}_p$ is a quasi-chain, $v_{\widehat{\mathcal{H}}_p} + 1 < w_{\widehat{\mathcal{H}}_p}$ and $\operatorname{fmesh}(p) < \min\{\frac{\varepsilon}{3}, 1\}$. We have $(p, v_{\widehat{\mathcal{H}}_p} + 1, w_{\widehat{\mathcal{H}}_p} - 1) \in \Delta$ and by Claim 6 and Lemma 25 there exist recursive sequences (l_k), (v_k) and (w_k) in $\mathbf{N}$ such that

$$l_0 = p, \ v_0 = v_{\widehat{\mathcal{H}}_p} + 1, \ w_0 = w_{\widehat{\mathcal{H}}_p} - 1, \tag{10}$$

$$(l_k, v_k, w_k) \in \Delta, \ \forall k \in \mathbf{N}, \tag{11}$$

$$(l_k, v_k, w_k, l_{k+1}, v_{k+1}, w_{k+1}) \in \Gamma,\ \forall k \in \mathbf{N}. \tag{12}$$

For $k \in \mathbf{N}$ let $C_0^k, \dots, C_{m_k}^k$ denotes the finite sequence $J_{(l_k)_{v_k}}, \dots, J_{(l_k)_{w_k}}$. By (11) and Claim 1, $C_0^k, \dots, C_{m_k}^k$ is a chain and each of its links intersects S, $\forall k \in \mathbf{N}$. It follows easily from (12) that $\mathrm{fmesh}(l_k) < 2^{-k}$, $\forall k \in \mathbf{N}$. Since $\mathrm{diam}(J_j) \le \mathrm{fdiam}(j)$, $\forall j \in \mathbf{N}$, we conclude from the definition of fmesh that $C_0^k, \dots, C_{m_k}^k$ is a $2^{-k}-$chain. Since $\overline{J_j} \subseteq \widehat{J}_j$, $\forall j \in \mathbf{N}$, $\overline{C_0^{k+1}}, \dots, \overline{C_{m_{k+1}}^{k+1}}$ strongly refines $C_0^k, \dots, C_{m_k}^k$, $\forall k \in \mathbf{N}$.

Let $K = \bigcap_{k\in\mathbf{N}} \left(\overline{C_0^{k+1}} \cup \ldots \cup \overline{C_{m_{k+1}}^{k+1}}\right)$. By Lemma 41 K is a continuum chainable from a' to b', where $a' \in \bigcap_{k\in\mathbf{N}} J_{(l_k)_0}$, $b' \in \bigcap_{k\in\mathbf{N}} J_{(l_k)_{\overline{l_k}}}$. It follows that a' and b' are recursive points in (X, d, α). Since for each $k \in \mathbf{N}$ the chain $C_0^k, \dots, C_{m_k}^k$ covers K and each of its links intersects K (Lemma 30), Lemma 14 implies that K is a recursive set in (X, d, α). Since for each $k \in \mathbf{N}$ each link of the chain $C_0^k, \dots, C_{m_k}^k$ intersects S, we have $K \subseteq \overline{S}$, hence $K \subseteq S$.

Finally, let us prove that $d(a, a') < \varepsilon$, $d(b, b') < \varepsilon$. We have $\mathrm{fmesh}(l_0) < \frac{\varepsilon}{3}$. It follows from (10) and Claim 1 that $C_{v_{\mathcal{H}_{l_0}}} \cap C_{v_0} \neq \emptyset$ which together with $C_{v_{\mathcal{H}_{l_0}}} \cap A \neq \emptyset$, $a \in A$, $a' \in C_{v_0}$ and the fact that $\mathrm{diam}(A), \mathrm{diam}(C_{v_{\mathcal{H}_{l_0}}}), \mathrm{diam}(C_{v_0}) < \frac{\varepsilon}{3}$ imply that $d(a, a') < \varepsilon$. Similarly $d(b, b') < \varepsilon$. □

Corollary 43. *Let (X, d, α) be a computable metric space which has the effective covering property and compact closed balls and let S be a co-r.e. arc in this space. Then for all $a, b \in S$ and $\varepsilon > 0$ there exist recursive points $a', b' \in S$, $a' \neq b'$, such that $d(a, a') < \varepsilon$, $d(b, b') < \varepsilon$ and such that the subarc of S determined by a' and b' is recursive.*

Theorem 44. *Let (X, d, α) be a computable metric space which has the effective covering property and compact closed balls. Let $S \subseteq X$ be a chainable and decomposable continuum. Suppose S is a co-r.e. set in (X, d, α). Then for each $\varepsilon > 0$ there exists a subcontinuum K of S which is recursive in (X, d, α) and such that $\rho(S, K) < \varepsilon$, where ρ is the Hausdorff metric. Moreover, K can be chosen so that it is chainable from a to b, where a and b are recursive points in S.*

Proof. Let A and B be proper subcontinua of S whose union is S. We claim that there exists $r > 0$ such that for each $r-$chain $C_0, \dots, C_m$ in X which covers S there exists $i \in \{0, \dots, m\}$ such that $C_i \cap A = \emptyset$. Indeed, if such r does not exist, then $S \subseteq \overline{A}$, hence $S \subseteq A$ which is impossible. Similar claim holds for B. Therefore there exists an $\varepsilon-$chain $C_0, \dots, C_m$ in X which covers S, such that $C_i \cap S \neq \emptyset$, $\forall i \in \{0, \dots, m\}$ and $C_v \cap A = \emptyset$, $C_w \cap B = \emptyset$ for some $v, w \in \{0, \dots, m\}$. It follows from Lemma 30 that ($C_0 \cap A = \emptyset$ or $C_m \cap A = \emptyset$) and ($C_0 \cap B = \emptyset$ or $C_m \cap B = \emptyset$).

Suppose $C_0 \cap A = \emptyset$. Then $C_0 \cap B \neq \emptyset$, $C_m \cap B = \emptyset$ and $C_m \cap A \neq \emptyset$. Let $b \in C_0 \cap B$, $a \in C_m \cap A$. Then $a \in A \setminus B$, $b \in B \setminus A$. Theorem 42 and the fact that C_0 and C_m are open sets imply that there exist recursive points a', b' in S such that $a' \in C_m$, $b' \in C_0$ and a recursive subcontinuum K of S chainable from a' to b'. We have $K \cap C_0 \neq \emptyset$, $K \cap C_m \neq \emptyset$ and Lemma 30 implies that $K \cap C_i \neq \emptyset$, $\forall i \in \{0, \dots, m\}$. If $s \in S$, then $s \in C_i$ for some $i \in \{0, \dots, m\}$ and since $C_i \cap K \neq \emptyset$ there exists $x \in K$ such that $d(s,x) < \operatorname{diam} C_i < \varepsilon$. It follows from Proposition 6 that $\rho(S,K) < \varepsilon$. We get the same conclusion in the case $C_m \cap A = \emptyset$. □

Let us summarize. Suppose X is a computable metric space which has the effective covering property and compact closed balls. If S is a co-recursively enumerable subset of X which, as a subspace of X, is a circularly chainable, but not chainable continuum, then S is recursive (Theorem 35). On the other hand, since each chainable continuum which is not circularly chainable is decomposable [Burgess 1959], we have a slightly weaker version of Theorem 44: if S is a co-r.e. subset of X which, as a subspace, is a chainable, but not circularly chainable continuum, then for each $\varepsilon > 0$ there exists a recursive subcontinuum of S which is $\varepsilon-$close to S.

Let us take now, for simplicity, that S is a co-r.e. subset of X which is neither indecomposable nor 2-indecomposable. Then Theorem 35 and Theorem 44 give:

1) if S is circularly chainable, then S is recursive;
2) if S is chainable, then S is "almost recursive".

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References

[Brattka and Presser 2003] Brattka, V., Presser, G.: "Computability on subsets of metric spaces"; Theoretical Computer Science 305 (2003) 43-76.

[Burgess 1959] Burgess, C.E.: "Chainable continua and indecomposability"; Pac. J. Math. 9, 1959, 653-659.

[Christenson and Voxman 1977] Christenson, C.O., Voxman, W.L.: "Aspects of Topology", Marcel Dekker, Inc., New York, 1977.

[Miller 2002] Miller, J.S., "Effectiveness for Embedded Spheres and Balls", Electronic Notes in Theoretical Computer Science, volume 66, Elsevier, 2002, 127-138.

[Nadler 1992] Nadler, S.B.: "Continuum theory"; Marcel Dekker, Inc., New York, 1992.

[Pour-El and Richards 1989] Pour-El, M.B., Richards, I., "Computability in Analysis and Physics", Springer-Verlag, Berlin-Heielberg-New York, 1989.

[Specker 1959] Specker, E., "Der Satz vom Maximum in der rekursiven Analysis", Constructivity in Mathematics (A. Heyting, ed.), North Holland Publ. Comp., Amsterdam 1959, 254-265.

[Weihrauch 2000] Weihrauch, K.: "Computable Analysis"; Springer, Berlin, 2000.