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Decision Making with Uncertainty Information Based on Lattice-Valued Fuzzy Concept Lattice

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Abstract: For the processing of decision making with uncertainty information, this paper establishes a decision model based on lattice-valued logic and researches the algorithm for extracting the maximum decision rules. Firstly, we further research the lattice-valued fuzzy concept lattice by combining the lattice implication algebra and classical concept lattice; secondly, we define the lattice-valued decision context as the equivalent form of decision information system and establish the single-target decision model and talk about some properties of the decision rules; finally, we give the calculating methods of decision rules with different decision values and the algorithm for extracting the maximum decision rules.

Keywords: Decision making, Uncertainty, Concept lattice, Lattice-valued logic, Decision rule
Categories: I.2.3

1 Introduction

In many cases, decision making processes generally consist of some uncertainty problems [Iraj, 08; Xu, 08]. The increasing complexity of environment makes it too difficult or too ill-defined to be amenable for description in conventional quantitative expressions. It is more suitable to provide their decision information by means of various expressions rather than numerical ones, such as linguistic values “less”, “better” [Delgado, 02; Herrera, 00, 05; Yang, 08a], or symbolic values “++”, “+” [Xu, 07]. In order to possess a reasonable structure for these different expressions, it has been the research hotspot to construct a decision model with a kind of information expression at present.

Up to now, several proposed decision models mainly include: (1) the decision model based on rough set [Zhang, 05], which acquires the rules through decision matrixes and calculates the minimal decision rules by reduction theory; (2) the decision model based on granular computing [Hobbs, 85; Zadeh, 97; Yao, 03, 04; Cheng, 07], which decomposes complex information into many simpler sorts according to the characteristic and capability of decision information; (3) the decision model based on classical concept lattice [Liang, 04; Zhang, 06; Wang, 07], which establishes the decision formal context according to the decision information and acquires the rules by constructing the decision concept lattice which consists of the antecedent concept lattice and consequent concept lattice. But these models constructed in the certainty environment are inappropriate to deal with uncertainty information. Especially, for the model (3), the necessary condition for utilizing it to extract the decision rules is to establish a surjection between the antecedent concept lattice and consequent concept lattice and the model (3) will not function properly if the decision formal context is not consistent, i.e., there doesn't exist a surjection. However, in most cases, not all the decision concept lattices can find the relevant surjection and the model (3) can't deal with the uncertainty information. Moreover, for the decision making with a kind of uncertainty information --- the fuzzy information, the general fuzzy concept lattice based on the fuzzy logic has no researches on it to this day.

Classical concept lattice as a conceptual clustering method was proposed by Wille in 1982 [Wille, 82; Ganter, 99], where an object-attribute view of data is developed. Due to the fact that the concept lattice can provides a theoretical framework for the design and discovery concept hierarchies from relational information system, this paper still utilizes a kind of concept lattice to deal with the uncertainty problems in decision making. This kind of concept lattice --- the lattice-valued fuzzy concept lattice researched in this paper is established based on the combination of lattice implication algebra and classical concept lattice [Yang, 08]. The introduction of the lattice-valued logic based on the lattice implication algebra into classical concept lattice is the important characteristic of the lattice-valued fuzzy concept lattice different from the general fuzzy concept lattice. For the non-numerical information, the processing ability of the lattice-valued fuzzy concept lattice is better than that of other concept lattices. Especially for decision making with uncertainty information, the lattice-valued fuzzy concept lattice can be looked as a useful mathematical tool.

About the lattice-valued logic introduced in this paper, Xu established it based on the lattice implications algebra [Xu, 93; 99] in 1993, which is an alternative approach to treat fuzziness and incomparability, and researched the corresponding reasoning theories and methods [Xu, 94; 00; 01].

The lattice-valued fuzzy concept lattice is used as the rules base, which is called the lattice-valued decision concept lattice, and its characteristics different from the general rules bases are mainly as follows: (1) all the decision rules in the same lattice-valued decision concept lattice have the same decision value, which is the essential difference from general rules bases. Such a rules base not only is beneficial to collect information and establish rules but also makes the extraction of potential rules simpler; (2) the antecedent of decision rules in the lattice-valued fuzzy concept lattice consists of two types of sets, which can depict the question more concretely; (3) the new decision rules can have their respective decision values according to the calculation of truth values, by which, the rules base can be enlarged into the one with different decision values.

In particularly, the single-target decision model established based on the lattice-valued fuzzy concept lattice is entirely different from the one based on classical concept lattice: on the one hand, we need only to construct one lattice-valued fuzzy concept lattice and don't need to construct antecedent concept lattice and consequent concept lattice on the structure form, which simplifies the decision model and dispenses with establishing the surjection on it; on the other hand, we introduce the calculating method of decision values, which provides more reasonable methods in decision making.

Based on these analyses, this paper puts forward the method of decision making with uncertainty information based on the lattice-valued fuzzy concept lattice. Section 2 gives an overview of some basic theories of the concept lattice and the lattice implication algebra. In Section 3, we briefly research the related works of the lattice-valued fuzzy concept lattice. Successively, we construct the single-target decision model in Section 4, where we firstly establish the lattice-valued single-target decision concept lattice and give the definition of decision rule; secondly, we present the calculating method of decision rules with different decision values and talk about the decision rules properties; finally, we discuss the algorithm of extracting the maximum decision rules from the lattice-valued decision context and establish the decision rules base. Concluding remarks and future researches are presented in Section 5.

2 Preliminaries

In this section, we review briefly the classical concept lattice and the lattice implication algebra and they are the foundations of constructing the lattice-valued fuzzy concept lattice.

Definition 2.1 [Birkhoff, 67] A partial ordered set is a set in which a binary relation $\leq$ is defined, which satisfies the following conditions: for any x, y, z,

(1) $x \leq x$, for any x (Reflexive);

(2) $x \leq y$ and $y \leq x$ implies $x = y$ (Antisymmetry);

(3) $x \leq y$ and $y \leq z$ implies $x \leq z$ (Transitivity).

Definition 2.2 [Birkhoff, 67] Let L be an arbitrary set, and let there be given two binary operations on L, denoted by $\wedge$ and $\vee$. Then the structure $(L, \wedge, \vee)$ is an algebraic structure with two binary operations. We call the structure $(L, \wedge, \vee)$ a lattice provided that it satisfies the following properties:

(1) For any $x, y, z \in L$, $x \wedge (y \wedge z) = (x \wedge y) \wedge z$ and $x \vee (y \vee z) = (x \vee y) \vee z$;

(2) For any $x, y \in L$, $x \wedge y = y \wedge x$ and $x \vee y = y \vee x$;

(3) For any $x \in L$, $x \wedge x = x$ and $x \vee x = x$;

(4) For any $x, y \in L$, $x \wedge (x \vee y) = x$ and $x \vee (x \wedge y) = x$.

Definition 2.3 [Xu, 03] If a lattice has a smallest element, denoted by O, and a greatest element, denoted by I, then it is called a bounded lattice.

Definition 2.4 [Xu, 93; 03] Let $(L, \wedge, \vee, O, I)$ be a bounded lattice with an order-reversing involution $'$, I and O the greatest and the smallest element of L, respectively, and $\rightarrow: L \times L \rightarrow L$ a mapping. $(L, \wedge, \vee, ', \rightarrow, O, I)$ is called a lattice implication algebra, if the following conditions hold for any $x, y, z \in L$:

(1) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$;

(2) $x \rightarrow x = I$;

(3) $x \to y = y' \to x'$;

(4) $x \to y = y \to x = I$ implies $x = y$;

(5) $(x \to y) \to y = (y \to x) \to x$;

(6) $(x \vee y) \to z = (x \to z) \wedge (y \to z)$;

(7) $(x \wedge y) \to z = (x \to z) \vee (y \to z)$.

Definition 2.5 [Xu, 03] A Wajsberg algebra is an algebra $(L, \to, *, 1)$ of type (2,1,0) such that

(1) $1 \to x = x$;

(2) $(x \to y) \to ((y \to z) \to (x \to z)) = 1$;

(3) $(x \to y) \to y = (y \to x) \to x$;

(4) $(x^* \to y^*) \to (y \to x) = 1$.

There is a one-to-one corresponding between Wajsberg algebras and MV-algebras [Höhle, 95; 95a].

Theorem 2.1 Let $(L, \wedge, \vee, ', \to, O, I)$ be a lattice implication algebra, define $' = *$, $I = 1$, then $(L, \wedge, \vee, *, \to, 1)$ is a Wajsberg algebra.

Proof.

$$\begin{aligned}(x \to x) = I &\Rightarrow I \to (x \to x) = I \\ &\Rightarrow x \to (I \to x) = I \\ &\Rightarrow x = I \to x \\ &\Rightarrow x = 1 \to x;\end{aligned}$$

$$\begin{aligned}(x \to y) \to ((y \to z) \to (x \to z)) &= (x \to y) \to (x \to ((y \to z) \to z)) \\ &= (x \to y) \to (x \to ((z \to y) \to y)) \\ &= (x \to y) \to ((z \to y) \to (x \to y)) \\ &= (z \to y) \to ((x \to y) \to (x \to y)) \\ &= (z \to y) \to I \\ &= 1;\end{aligned}$$

$$\begin{aligned}x \to y = y' \to x' &\Rightarrow (y' \to x') \to (x \to y) = I \\ &\Rightarrow (x' \to y') \to (y \to x) = I \\ &\Rightarrow (x^* \to y^*) \to (y \to x) = 1. \quad \square\end{aligned}$$

Example 2.1 [Xu, 03] (**Boolean Algebra**) Let $(L, \wedge, \vee, ')$ be a Boolean lattice, for any $x, y \in L$, define

$$x \to y = x' \vee y,$$

then $(L, \wedge, \vee, ', \to, 0, 1)$ is a lattice implication algebra.

Example 2.2 [Xu, 03] **(Lukasiewicz implication algebra on [0,1])** If the operations on [0,1] are defined respectively as follows:

$$x \vee y = \max\{x, y\},$$

$$x \wedge y = \min\{x, y\},$$

$$x' = 1 - x,$$

$$x \to y = \min\{1, 1 - x + y\},$$

then $([0,1], \vee, \wedge, ', \to, 0, 1)$ is a lattice implication algebra.

Definition 2.6 [Ganter, 99] A formal context in classical concept lattice is defined as a set structure (G, M, I) consisting of the finite sets G and M and a binary relation $I \subseteq G \times M$. The elements of G and M are called objects and attributes, respectively, and the relationship gIm is read: the object *g* has the attribute *m*. For a set of objects $A \subseteq G$, and a set of attributes $B \subseteq M$, A^* is defined as the set of features shared by all the objects in *A*, B^* is defined as the set of objects that posses all the features in *B*, that is,

$$A^* = \{m \in M \mid gIm \, \forall g \in A\}, \quad B^* = \{g \in G \mid gIm \, \forall m \in B\}.$$

Definition 2.7 [Ganter, 99] A formal concept of the context (G, M, I) is defined as a pair (A, B) with $A \subseteq G$, $B \subseteq M$ and $A^* = B$, $B^* = A$. The set *A* is called the extent and *B* the intent of the concept (A, B).

3 Lattice-Valued Fuzzy Concept Lattice

In order to provide a mathematical tool for incomparability fuzzy information processing, we further research the lattice-valued fuzzy concept lattice, which is the combination of classical concept lattice and lattice implication algebra and totally different from the general fuzzy concept lattice. Its ideological core is constructing the lattice-valued fuzzy relation between objects and attributes. In this section, we study the definitions and properties of the lattice-valued fuzzy concept lattice and give an example to illustrate it.

Definition 3.1 [Yang, 08] A 4-tuple $K = (G, M, L, \tilde{I})$ is called lattice-valued fuzzy context, where $G = \{g_1, g_2, \cdots, g_p\}$ is the non-empty finite objects set, $M = \{m_1, m_2, \cdots, m_q\}$ is the non-empty finite attributes set, $(L, \wedge, \vee, ', \to, O, I)$ is a lattice implication algebra, $\tilde{I}$ is a fuzzy relation between G and M, i.e., $\tilde{I}: G \times M \to L$.

Let G be a non-empty finite objects set and $(L, \wedge, \vee, ', \to, O, I)$ be a lattice implication algebra. Denote the set of all the fuzzy subsets on G as L^G, for any $\tilde{A}_1, \tilde{A}_2 \in L^G$ and $g \in G$, $\tilde{A}_1 \subseteq \tilde{A}_2 \Leftrightarrow \tilde{A}_1(g) \le \tilde{A}_2(g)$, then $(L^G, \subseteq)$ is a partial ordered set.

Let M be a non-empty finite attributes set and $(L, \wedge, \vee, ', \to, O, I)$ be a lattice implication algebra. Denote the set of all the fuzzy subsets on M as L^M, for any $\tilde{B}_1, \tilde{B}_2 \in L^M$ and $m \in M$, $\tilde{B}_1 \subseteq \tilde{B}_2 \Leftrightarrow \tilde{B}_1(m) \le \tilde{B}_2(m)$, then $(L^M, \subseteq)$ is a partial ordered set.

Theorem 3.1 [Yang, 08] Let $K = (G, M, L, \tilde{I})$ be a lattice-valued fuzzy context and L a lattice implication algebra, define mappings f, h between L^G and L^M,

$$\begin{cases} f: L^G \to L^M,\ f\left(\tilde{A}\right)(m) = \underset{g\in G}{\wedge}\left(\tilde{A}(g) \to \tilde{I}(g,m)\right) \\ h: L^M \to L^G,\ h\left(\tilde{B}\right)(g) = \underset{m\in M}{\wedge}\left(\tilde{B}(m) \to \tilde{I}(g,m)\right) \end{cases},$$

then for any $\tilde{A}\in L^G$, $\tilde{B}\in L^M$, (f,h) is a Galois connection.

Definition 3.2 [Yang, 08] A lattice-valued fuzzy concept of $K=(G,M,L,\tilde{I})$ is defined as a pair $(\tilde{A},\tilde{B})$ with $\tilde{A}\in L^G$, $\tilde{B}\in L^M$ and $f(\tilde{A})=\tilde{B}$, $h(\tilde{B})=\tilde{A}$. For any lattice-valued fuzzy concepts $(\tilde{A}_1,\tilde{B}_1)$ and $(\tilde{A}_2,\tilde{B}_2)$, define $(\tilde{A}_1,\tilde{B}_1)\le(\tilde{A}_2,\tilde{B}_2)\Leftrightarrow \tilde{A}_1\subseteq\tilde{A}_2(or\ \tilde{B}_2\subseteq\tilde{B}_1)$ and denote the set $\mathbf{C}(K)=\left\{(\tilde{A},\tilde{B})\,\middle|\, f(\tilde{A})=\tilde{B}, h(\tilde{B})=\tilde{A}\right\}$ be the lattice-valued fuzzy concept lattice.

Theorem 3.2 [Yang, 08] Let $K=(G,M,L,\tilde{I})$ be a lattice-valued fuzzy context, and (f,h) the Galois connection on it, for any $\tilde{A}_1,\tilde{A}_2,\tilde{A}\in L^G$, $\tilde{B}_1,\tilde{B}_2,\tilde{B}\in L^M$, there are following properties:

(1) $\tilde{A}_1\subseteq\tilde{A}_2\Rightarrow f(\tilde{A}_2)\subseteq f(\tilde{A}_1)$, $\tilde{B}_1\subseteq\tilde{B}_2\Rightarrow h(\tilde{B}_2)\subseteq h(\tilde{B}_1)$;

(2) $\tilde{A}\subseteq hf(\tilde{A})$, $\tilde{B}\subseteq fh(\tilde{B})$;

(3) $f(\tilde{A})=fhf(\tilde{A})$, $h(\tilde{B})=hfh(\tilde{B})$;

(4) $f(\tilde{A}_1\cup\tilde{A}_2)=f(\tilde{A}_1)\cap f(\tilde{A}_2)$, $h(\tilde{B}_1\cup\tilde{B}_2)=h(\tilde{B}_1)\cap h(\tilde{B}_2)$.

Example 3.1 Let us consider a lattice-valued fuzzy context $K=(G,M,L,\tilde{I})$ depicted in Table 1, where $G=\{g_1,g_2\}$, $M=\{m_1,m_2,m_3,m_4\}$, $(L,\wedge,\vee,',\to,O,I)$ is a lattice implication algebra and its Hasse diagram is shown as Figure 1 and implication operator as Table 2:

$\tilde{I}$	m_1	m_2	m_3	m_4
g_1	a	b	c	I
g_2	d	c	O	b

Table 1: Lattice-valued fuzzy context $(G,M,L,\tilde{I})$

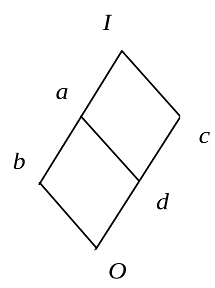

Figure 1: Hasse diagram of $L_6=\{O,a,b,c,d,I\}$

$\to$	I	a	b	c	d	O
I	I	a	b	c	d	O
a	I	I	a	c	c	d
b	I	I	I	c	c	c
c	I	a	b	I	a	b
d	I	I	a	I	I	a
O	I	I	I	I	I	I

Table 2: Implication Operator of $L_6 = \{O, a, b, c, d, I\}$

For Table 1, we can calculate its lattice-valued fuzzy concepts according to the Definition 3.2 and Theorem 3.1 as follows:
$C_1 = (II, dOOb)$, $C_2 = (aI, ddOb)$, $C_3 = (Ia, dOda)$, $C_4 = (Ic, abOb)$,
$C_5 = (bI, dcOb)$, $C_6 = (aa, cdda)$, $C_7 = (Ib, dOcI)$, $C_8 = (Id, abda)$,
$C_9 = (ac, aaOb)$, $C_{10} = (cc, abbb)$, $C_{11} = (ba, ccda)$, $C_{12} = (ab, cdcI)$,
$C_{13} = (ad, Iada)$, $C_{14} = (bc, aIOb)$, $C_{15} = (IO, abcI)$, $C_{16} = (cd, abaa)$,
$C_{17} = (dc, aabb)$, $C_{18} = (bb, cccI)$, $C_{19} = (aO, IacI)$, $C_{20} = (bd, IIda)$,
$C_{21} = (cO, abII)$, $C_{22} = (dd, Iaaa)$, $C_{23} = (Oc, aIbb)$, $C_{24} = (bO, IIcI)$,
$C_{25} = (dO, IaII)$, $C_{26} = (Od, IIaa)$, $C_{27} = (OO, IIII)$.

In the above lattice-valued fuzzy concepts, we illustrate the calculation method by the fuzzy concept $C_2 = (aI, ddOb)$ as follows:

When $\tilde{A} = \{a, I\}$, we will firstly calculate
$f(\tilde{A}) = \{f(\tilde{A})(m_1), f(\tilde{A})(m_2), f(\tilde{A})(m_3), f(\tilde{A})(m_4)\}$ by Theorem 3.1, in which,

$$f(\tilde{A})(m_1) = \wedge_{g\in G}(\tilde{A}(g) \to \tilde{I}(g, m_1)) = (a \to a) \wedge (I \to d) = I \wedge d = d,$$

$$f(\tilde{A})(m_2) = \wedge_{g\in G}(\tilde{A}(g) \to \tilde{I}(g, m_2)) = (a \to b) \wedge (I \to c) = a \wedge c = d,$$

$$f(\tilde{A})(m_3) = \wedge_{g\in G}(\tilde{A}(g) \to \tilde{I}(g, m_3)) = (a \to c) \wedge (I \to O) = c \wedge O = O,$$

$$f(\tilde{A})(m_4) = \wedge_{g\in G}(\tilde{A}(g) \to \tilde{I}(g, m_4)) = (a \to I) \wedge (I \to b) = I \wedge b = b,$$

i.e., $f(\tilde{A}) = \{d, d, O, b\}$ denoted by $\tilde{B} = \{d, d, O, b\}$;

We will secondly calculate
$h(\tilde{B}) = \{h(\tilde{B})(g_1), h(\tilde{B})(g_2)\}$ by Theorem 3.1, in which,

$$h(\tilde{B})(g_1) = \wedge_{m\in M}(\tilde{B}(m) \to \tilde{I}(g_1, m)) = (d \to a) \wedge (d \to b) \wedge (O \to c) \wedge (b \to I) = a,$$

$h(\tilde{B})(g_2) = \underset{m \in M}{\wedge} (\tilde{B}(m) \to \tilde{I}(g_2, m)) = (d \to d) \wedge (d \to c) \wedge (O \to O) \wedge (b \to b) = I$,

i.e., $h(\tilde{B}) = \{a, I\} = \tilde{A}$, so $C_2 = (aI, ddOb)$ is a lattice-valued fuzzy concept.

All the above lattice-valued fuzzy concepts can be constructed into a concept lattice by Definition 3.2 and its Hasse diagram as Figure 2:

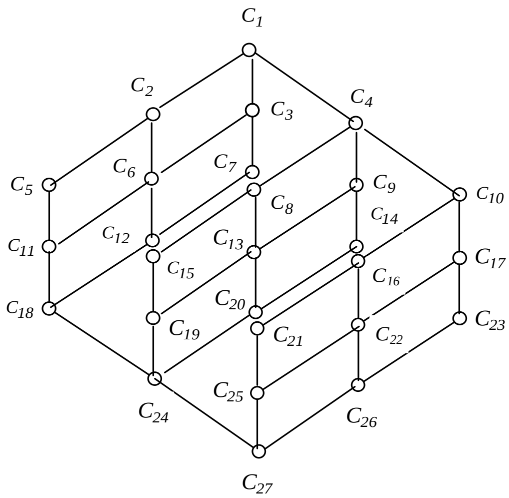

Figure 2: Hasse diagram of $\mathbf{C}(K)$

4 Decision Making Based on Lattice-Valued Fuzzy Concept Lattice

4.1 Decision Making with Uncertainty Information

In this section, we mainly define the lattice-valued decision context according to the concrete decision information system and establish the lattice-valued decision concept lattice as the single-target decision model.

Definition 4.1 A 4-tuple $K = (G, M \cup \{d\}, L, \tilde{I})$ is called a lattice-valued single-target decision context, where G is the non-empty finite objects set, M is the non-empty finite attributes set and d is a decision attribute, $(L, \wedge, \vee, ', \to, O, I)$ is a lattice implication algebra, $\tilde{I}$ is a relation between G and $M \cup \{d\}$, i.e., $\tilde{I} : G \times M \cup \{d\} \to L$.

Definition 4.2 Let $K = (G, M \cup \{d\}, L, \tilde{I})$ be a lattice-valued single-target decision context, a 4-tuple $K|_\alpha = (G_\alpha, M \cup \{d\}, L, \tilde{I}_\alpha)$ is called a lattice-valued single-target decision context with decision value α , where $G_\alpha = \{g \mid g\tilde{I}_\alpha d = \alpha, \alpha \in L\} \subseteq G$,

$\tilde{I}_\alpha : G_\alpha \times M \to L$, accordingly, $\boldsymbol{C}(K|_\alpha)$ is called a lattice-valued single-target decision concept lattice with decision value α .

Definition 4.3 Let $K|_\alpha = (G_\alpha, M \cup \{d\}, L, \tilde{I}_\alpha)$ be a lattice-valued single-target decision context with decision value α, for any $\tilde{A} \in L^{G_\alpha}$ and $\tilde{B} \in L^M$, define the Galois connection (f_α, h_α) between L^{G_α} and L^M as follows:

$$\begin{cases} f_\alpha : L^{G_\alpha} \to L^M, \ f_\alpha(A)(m) = \underset{g \in G_\alpha}{\wedge} \left(A(g) \to \tilde{I}_\alpha(g,m) \right) \\ h_\alpha : L^M \to L^{G_\alpha}, \ h_\alpha(B)(g) = \underset{m \in M}{\wedge} \left(B(m) \to \tilde{I}_\alpha(g,m) \right) \end{cases}.$$

Let be a lattice-valued single-target decision context with decision value α, denote:

$$\boldsymbol{C}_{G_\alpha}(K|_\alpha) = \left\{ \tilde{A} \middle| (\tilde{A},\tilde{B}) \in \mathbf{C}(K|_\alpha) \right\}, \ \boldsymbol{C}_M(K|_\alpha) = \left\{ \tilde{B} \middle| (\tilde{A},\tilde{B}) \in \mathbf{C}(K|_\alpha) \right\}.$$

In the following, “//” is expressed as the incomparability.

Definition 4.4 Let $K|_\alpha = (G_\alpha, M \cup \{d\}, L, \tilde{I}_\alpha)$ be a lattice-valued single-target decision context with decision value α , and $\boldsymbol{C}(K|_\alpha)$ the lattice-valued single-target decision concept lattice with decision value α, and (f_α, h_α) the Galois connection, the decision rule is defined as “ $(\tilde{A},\tilde{B}) \to \{d(\tilde{A},\tilde{B}) = \alpha\}$ ”, i.e., “ if $\tilde{A}$ and $\tilde{B}$, then the decision value of $(\tilde{A},\tilde{B})$ is α ”, if $\tilde{A}$ and $\tilde{B}$ satisfy at least one of the conditions as follows:

(1) $\tilde{B} = f_\alpha(\tilde{A})$ or $\tilde{B} // f_\alpha(\tilde{A})$, for any $\tilde{A} \in \boldsymbol{C}_{G_\alpha}(K|_\alpha)$, $\tilde{B} \in L^M$;

(2) $\tilde{A} = h_\alpha(\tilde{B})$ or $\tilde{A} // h_\alpha(\tilde{B})$, for any $\tilde{B} \in \boldsymbol{C}_M(K|_\alpha)$, $\tilde{A} \in L^{G_\alpha}$.

For the decision rules mined from the lattice-valued single-target decision context, it is necessary for us to find out the redundant rules and the matrix rules existed in this model.

Definition 4.5 Let $K|_\alpha = (G_\alpha, M \cup \{d\}, L, \tilde{I}_\alpha)$ be a lattice-valued single-target decision context with decision value α , and $\boldsymbol{C}(K|_\alpha)$ the lattice-valued single-target decision concept lattice with decision value α, and (f_α, h_α) the Galois connection, the decision rule “ $(\tilde{A},\tilde{B}) \to \{d(\tilde{A},\tilde{B}) = \alpha\}$ ” is called as the redundant rule, if $\tilde{A}$ and $\tilde{B}$ satisfy at least one of the conditions as follows:

(1) $\tilde{B} // f_\alpha(\tilde{A})$, for any $\tilde{A} \in \boldsymbol{C}_{G_\alpha}(K|_\alpha)$, $\tilde{B} \in L^M$;

(2) $\tilde{A} // h_\alpha(\tilde{B})$, for any $\tilde{B} \in \boldsymbol{C}_M(K|_\alpha)$, $\tilde{A} \in L^{G_\alpha}$.

Definition 4.6 Let $K|_\alpha = (G_\alpha, M \cup \{d\}, L, \tilde{I}_\alpha)$ be a lattice-valued single-target decision context with decision value α , and $\boldsymbol{C}(K|_\alpha)$ the lattice-valued single-target decision concept lattice with decision value α, for any $\tilde{A} \in L^{G_\alpha}$ and $\tilde{B} \in L^M$, the decision rule “ $(\tilde{A},\tilde{B}) \to \{d(\tilde{A},\tilde{B}) = \alpha\}$ ” is called as the maximum rule, if

$(\tilde{A},\tilde{B})\in \boldsymbol{C}(K|_{\alpha})$.

In this decision model, we can get not only the decision rules with decision value α but also the decision rules with decision values different from α according to the following two definitions.

Let $(L,\vee,\wedge,',\rightarrow,O,I)$ be a lattice implication algebra, we denote:

$$L^{>\alpha}=\{\beta|\ \beta>\alpha,\ \beta\in L\}$$

$$L^{<\alpha}=\{\beta|\ \beta<\alpha,\ \beta\in L\}$$

Definition 4.7 Let $K|_{\alpha}=(G_{\alpha},M\cup\{d\},L,\tilde{I}_{\alpha})$ be a lattice-valued single-target decision context with decision value α, and $\boldsymbol{C}(K|_{\alpha})$ the lattice-valued single-target decision concept lattice with decision value α, and (f_{α},h_{α}) the Galois connection, for any $\tilde{A}\in L^{G_{\alpha}}$ and $\tilde{B}\in L^{M}$, the decision rule is defined as "$(\tilde{A},\tilde{B})\rightarrow\{d(\tilde{A},\tilde{B})\in L^{>\alpha}\}$", i.e., " if $\tilde{A}$ and $\tilde{B}$, then the decision value of $(\tilde{A},\tilde{B})$ is more than α ", if $\tilde{A}$ and $\tilde{B}$ satisfy at least one of the conditions as follows:

(1) $\tilde{A}\in \boldsymbol{C}_{G_{\alpha}}(K|_{\alpha})$ and $\tilde{B}\supset f_{\alpha}(\tilde{A})$;

(2) $\tilde{B}\in \boldsymbol{C}_{M}(K|_{\alpha})$ and $\tilde{A}\supset h_{\alpha}(\tilde{B})$.

Definition 4.8 Let $K|_{\alpha}=(G_{\alpha},M\cup\{d\},L,\tilde{I}_{\alpha})$ be a lattice-valued single-target decision context with decision value α, and $\boldsymbol{C}(K|_{\alpha})$ the lattice-valued single-target decision concept lattice with decision value α, and (f_{α},h_{α}) the Galois connection, for any $\tilde{A}\in L^{G_{\alpha}}$ and $\tilde{B}\in L^{M}$, the decision rule is defined as "$(\tilde{A},\tilde{B})\rightarrow\{d(\tilde{A},\tilde{B})\in L^{<\alpha}\}$", i.e., " if $\tilde{A}$ and $\tilde{B}$, then the decision value of $(\tilde{A},\tilde{B})$ is less than α ", if $\tilde{A}$ and $\tilde{B}$ satisfy at least one of the conditions as follows:

(1) $\tilde{A}\in \boldsymbol{C}_{G_{\alpha}}(K|_{\alpha})$ and $\tilde{B}\subset f_{\alpha}(\tilde{A})$;

(2) $\tilde{B}\in \boldsymbol{C}_{M}(K|_{\alpha})$ and $\tilde{A}\subset h_{\alpha}(\tilde{B})$.

For the decision rules with decision values different from α, their concrete decision values can be calculated by the following definitions and theorems.

Definition 4.9 Let $(G,M,L,\tilde{I})$ be a lattice-valued fuzzy context, $\forall\tilde{A}_1,\tilde{A}_2\in L^{G_{\alpha}}$ and $\tilde{A}_2\subseteq\tilde{A}_1$, define the degree to which $\tilde{A}_1$ is more than $\tilde{A}_2$ as:

$$S(\tilde{A}_1-\tilde{A}_2)=\bigvee_{g\in G}\left(\tilde{A}_1(g)\rightarrow\tilde{A}_2(g)\right)';$$

$\forall\tilde{B}_1,\tilde{B}_2\in L^{M}$ and $\tilde{B}_2\subseteq\tilde{B}_1$, define the degree to which $\tilde{B}_1$ is more than $\tilde{B}_2$ as:

$$S(\tilde{B}_1-\tilde{B}_2)=\bigvee_{m\in M}\left(\tilde{B}_1(m)\rightarrow\tilde{B}_2(m)\right)'.$$

Theorem 4.1 Let $(G,M,L,\tilde{I})$ be a lattice-valued fuzzy context, then

(1) $S(\tilde{A}_2-\tilde{A}_3)\le S(\tilde{A}_1-\tilde{A}_3)$, if $\tilde{A}_3\subseteq\tilde{A}_2\subseteq\tilde{A}_1$, for any $\tilde{A}_1,\tilde{A}_2,\tilde{A}_3\in L^{G}$;

(2) $S(\tilde{B}_2 - \tilde{B}_3) \le S(\tilde{B}_1 - \tilde{B}_3)$, if $\tilde{B}_3 \subseteq \tilde{B}_2 \subseteq \tilde{B}_1$, for any $\tilde{B}_1, \tilde{B}_2, \tilde{B}_3 \in L^M$.

Proof. (1) $\forall \tilde{A}_1, \tilde{A}_2, \tilde{A}_3 \in L^G$ and $\tilde{A}_3 \subseteq \tilde{A}_2 \subseteq \tilde{A}_1$, for any $g \in G$, we can get $\tilde{A}_1(g) \to \tilde{A}_3(g) \le \tilde{A}_2(g) \to \tilde{A}_3(g)$,

$$S(\tilde{A}_1 - \tilde{A}_3) = \vee_{g\in G}\left(\tilde{A}_1(g) \to \tilde{A}_3(g)\right)' \ge \vee_{g\in G}\left(\tilde{A}_2(g) \to \tilde{A}_3(g)\right)' = S(\tilde{A}_2 - \tilde{A}_3).$$

(2) can be proved similarly. □

Definition 4.10 Let $K|_\alpha = (G_\alpha, M \cup \{d\}, L, \tilde{I}_\alpha)$ be a lattice-valued single-target decision context with decision value α, and (f_α, h_α) the Galois connection, for any decision rule " $(\tilde{A}, \tilde{B}) \to \left\{d(\tilde{A}, \tilde{B}) \in L^{>\alpha}\right\}$ " satisfying $\tilde{A} \in \boldsymbol{C}_{G_\alpha}(K|_\alpha)$ and $\tilde{B} \supset f_\alpha(\tilde{A})$, then $d(\tilde{A}, \tilde{B})$ can be calculated as:

$$d(\tilde{A}, \tilde{B}) = \alpha \oplus S\left(\tilde{B} - f_\alpha(\tilde{A})\right).$$

Definition 4.11 Let $K|_\alpha = (G_\alpha, M \cup \{d\}, L, \tilde{I}_\alpha)$ be a lattice-valued single-target decision context with decision value α, and (f_α, h_α) the Galois connection, for any decision rule " $(\tilde{A}, \tilde{B}) \to \left\{d(\tilde{A}, \tilde{B}) \in L^{>\alpha}\right\}$ " satisfying $\tilde{B} \in \boldsymbol{C}_M(K|_\alpha)$ and $\tilde{A} \supset h_\alpha(\tilde{B})$, then $d(\tilde{A}, \tilde{B})$ can be calculated as:

$$d(\tilde{A}, \tilde{B}) = \alpha \oplus S\left(\tilde{A} - h_\alpha(\tilde{B})\right).$$

Similarly, for the decision rules with the decision values less than α, the following Definition 4.12 and Definition 4.13 also show that the size degree of the two L-fuzzy attribute subsets and the size degree of the two L-fuzzy object subsets can be looked as the decrease degree of α .

Definition 4.12 Let $K|_\alpha = (G_\alpha, M \cup \{d\}, L, \tilde{I}_\alpha)$ be a lattice-valued single-target decision context with decision value α, and (f_α, h_α) the Galois connection, for any decision rule " $(\tilde{A}, \tilde{B}) \to \left\{d(\tilde{A}, \tilde{B}) \in L^{<\alpha}\right\}$ " satisfying $\tilde{A} \in \boldsymbol{C}_{G_\alpha}(K|_\alpha)$ and $\tilde{B} \subset f_\alpha(\tilde{A})$, then $d(\tilde{A}, \tilde{B})$ can be calculated as:

$$d(\tilde{A}, \tilde{B}) = \alpha \otimes S'\left(f_\alpha(\tilde{A}) - \tilde{B}\right).$$

Definition 4.13 Let $K|_\alpha = (G_\alpha, M \cup \{d\}, L, \tilde{I}_\alpha)$ be a lattice-valued single-target decision context with decision value α, and (f_α, h_α) the Galois connection, for any decision rule " $(\tilde{A}, \tilde{B}) \to \left\{d(\tilde{A}, \tilde{B}) \in L^{<\alpha}\right\}$ " satisfying $\tilde{B} \in \boldsymbol{C}_M(K|_\alpha)$ and $\tilde{A} \subset h_\alpha(\tilde{B})$, then $d(\tilde{A}, \tilde{B})$ can be calculated as:

$$d(\tilde{A}, \tilde{B}) = \alpha \otimes S'\left(h_\alpha(\tilde{B}) - \tilde{A}\right).$$

In the following, we will talk about the properties of the formulas, such as coherent, isotone [Zhang, 05].

Theorem 4.2 The formulas defined by Definition 4.10 and 4.11 are coherent.

Proof. We need only to prove that $\forall \tilde{A}_1, \tilde{A}_2 \in L^{G_\alpha}$, $\tilde{B}_1, \tilde{B}_2 \in L^M$, $d(\tilde{A}_2, \tilde{B}_2) = \alpha$, when $\tilde{A}_2 = \tilde{A}_1$ and $\tilde{B}_2 = \tilde{B}_1$, it is obviously that $S(\tilde{A}_2 - \tilde{A}_1) = S(\tilde{B}_2 - \tilde{B}_1) = O$, so $d(\tilde{A}_2, \tilde{B}_2) = \alpha$. □

Theorem 4.3 The formulas defined by Definition 4.12 and 4.13 are coherent.

Proof. The proof is similar to the one of Theorem 4.2. □

Theorem 4.4 The formulas defined by Definition 4.10 and 4.11 are isotone.

Proof. We firstly prove that the formula defined by Definition 4.10 is isotone, i.e., $\forall \tilde{A}, \tilde{A}_1, \tilde{A}_2 \in L^{G_\alpha}$, $\tilde{B}, \tilde{B}_1, \tilde{B}_2 \in L^M$, $(\tilde{A}, \tilde{B}) \to \{d(\tilde{A}, \tilde{B}) = \alpha\}$ and $\tilde{B}_2 = \tilde{B}_1 = \tilde{B}$, $\tilde{A} \subseteq \tilde{A}_1 \subseteq \tilde{A}_2$, it follows that

$$\begin{aligned} d(\tilde{A}_2, \tilde{B}_2) &= \alpha \oplus S(\tilde{A}_2 - \tilde{A}) \\ &= \alpha' \to S(\tilde{A}_2 - \tilde{A}) \\ &\geq \alpha' \to S(\tilde{A}_1 - \tilde{A}) \\ &= \alpha \oplus S(\tilde{A}_1 - \tilde{A}) \\ &= d(\tilde{A}_1, \tilde{B}_1)\,; \end{aligned}$$

The isotone of the fomular defined by Definition 4.11 can be proved similarly.

So, the formulas defined by Definition 4.10 and 4.11 are isotone. □

Theorem 4.5 The formulas defined by Definition 4.12 and 4.13 are isotone.

Proof. We firstly prove that the formula defined by Definition 4.12 is isotone, i.e., $\forall \tilde{A}, \tilde{A}_1, \tilde{A}_2 \in L^{G_\alpha}$, $\tilde{B}, \tilde{B}_1, \tilde{B}_2 \in L^M$, $(\tilde{A}, \tilde{B}) \to d(\tilde{A}, \tilde{B}) = \alpha$ and $\tilde{B}_2 = \tilde{B}_1 = \tilde{B}$, $\tilde{A}_2 \subseteq \tilde{A}_1 \subseteq \tilde{A}$, it follows that

$$\begin{aligned} d(\tilde{A}_2, \tilde{B}_2) &= \alpha \otimes S'(\tilde{A} - \tilde{A}_2) \\ &= \left(\alpha \to S(\tilde{A} - \tilde{A}_2)\right)' \\ &\leq \left(\alpha \to S(\tilde{A} - \tilde{A}_1)\right)' \\ &= \alpha \otimes \pi'(\tilde{A} \subseteq \tilde{A}_1) \\ &= d(\tilde{A}_1, \tilde{B}_1)\,; \end{aligned}$$

The isotone of the formula defined by Definition 4.13 can be proved similarly.

So, the formulas defined by Definition 4.12 and 4.13 are isotone. □

4.2 Properties of decision rules

The properties of decision rules will play a certain extent role on mining all the decision rules, so we talk about them in the following.

Theorem 4.6 Let $K|_\alpha = (G_\alpha, M \cup \{d\}, L, \tilde{I}_\alpha)$ be a lattice-valued single-target decision context with decision value α. If $\forall \tilde{A}, \tilde{A}_1, \tilde{A}_2 \in L^{G_\alpha}$, $\exists \tilde{B} \in L^M$, s.t., "$(\tilde{A}, \tilde{B}) \to \{d(\tilde{A}, \tilde{B}) = \alpha\}$" is a maximum decision rule and $d(\tilde{A}_1, \tilde{B}) \in L^{>\alpha}$, $d(\tilde{A}_2, \tilde{B}) \in L^{>\alpha}$, then

(1) $d(\tilde{A}_1 \cup \tilde{A}_2, \tilde{B}) = d(\tilde{A}_1, \tilde{B}) \vee d(\tilde{A}_2, \tilde{B})$;

(2) $d(\tilde{A}_1 \cap \tilde{A}_2, \tilde{B}) \le d(\tilde{A}_1, \tilde{B}) \wedge d(\tilde{A}_2, \tilde{B})$.

Proof. (1) $\forall \tilde{A}, \tilde{A}_1, \tilde{A}_2 \in L^{G_\alpha}$, $\tilde{B}_2 \in L^M$,

$$
\begin{aligned}
d(\tilde{A}_1 \cup \tilde{A}_2, \tilde{B}) &= \alpha \oplus S(\tilde{A}_1 \cup \tilde{A}_2 - \tilde{A}) \\
&= \alpha' \to S(\tilde{A}_1 \cup \tilde{A}_2 - \tilde{A}) \\
&= \alpha' \to \bigvee_{g \in G} \left((\tilde{A}_1 \vee \tilde{A}_2)(g) \to \tilde{A}(g) \right)' \\
&= \alpha' \to \left(\bigwedge_{g \in G} \left(\tilde{A}_1(g) \to \tilde{A}(g) \right) \wedge \bigwedge_{g \in G} \left(\tilde{A}_2(g) \to \tilde{A}(g) \right) \right)' \\
&= \left(\alpha' \to \bigvee_{g \in G} \left(\tilde{A}_1(g) \to \tilde{A}(g) \right)' \right) \vee \left(\alpha' \to \bigvee_{g \in G} \left(\tilde{A}_2(g) \to \tilde{A}(g) \right)' \right) \\
&= \left(\alpha \oplus S(\tilde{A}_1 - \tilde{A}) \right) \vee \left(\alpha \oplus S(\tilde{A}_2 - \tilde{A}) \right) \\
&= d(\tilde{A}_1, \tilde{B}) \vee d(\tilde{A}_2, \tilde{B});
\end{aligned}
$$

(2) $d(\tilde{A}_1 \cap \tilde{A}_2, \tilde{B})$

$$
\begin{aligned}
&= \alpha \oplus S(\tilde{A}_1 \cap \tilde{A}_2 - \tilde{A}) \\
&= \alpha' \to S(\tilde{A}_1 \cap \tilde{A}_2 - \tilde{A}) \\
&= \alpha' \to \bigvee_{g \in G_\alpha} \left((\tilde{A}_1 \cap \tilde{A}_2)(g) \to \tilde{A}(g) \right)' \\
&= \alpha' \to \bigvee_{g \in G_\alpha} \left(\left(\tilde{A}_1(g) \to \tilde{A}(g) \right) \vee \left(\tilde{A}_2(g) \to \tilde{A}(g) \right) \right)' \\
&\le \alpha' \to \left(\bigwedge_{g \in G_\alpha} \left(\tilde{A}_1(g) \to \tilde{A}(g) \right) \vee \bigwedge_{g \in G_\alpha} \left(\tilde{A}_2(g) \to \tilde{A}(g) \right) \right)' \\
&= \left(\alpha' \to \bigvee_{g \in G_\alpha} \left(\tilde{A}_1(g) \to \tilde{A}(g) \right)' \right) \wedge \left(\alpha' \to \bigvee_{g \in G_\alpha} \left(\tilde{A}_2(g) \to \tilde{A}(g) \right)' \right) \\
&= \left(\alpha' \to S(\tilde{A}_1 - \tilde{A}) \right) \wedge \left(\alpha' \to S(\tilde{A}_1 - \tilde{A}) \right) \\
&= \left(\alpha \oplus S(\tilde{A}_1 - \tilde{A}) \right) \wedge \left(\alpha \oplus S(\tilde{A}_2 - \tilde{A}) \right) \\
&= d(\tilde{A}_1, \tilde{B}) \wedge d(\tilde{A}_2, \tilde{B}). \quad \square
\end{aligned}
$$

Theorem 4.7 Let $K|_\alpha = (G_\alpha, M \cup \{d\}, L, \tilde{I}_\alpha)$ be a lattice-valued single-target decision context with decision value α. If $\forall \tilde{A}, \tilde{A}_1, \tilde{A}_2 \in L^{G_\alpha}$, $\exists \tilde{B} \in L^M$, s.t., " $(\tilde{A}, \tilde{B}) \to \{d(\tilde{A}, \tilde{B}) = \alpha\}$ " is a maximum decision rule and $d(\tilde{A}_1, \tilde{B}) \in L^{<\alpha}$, $d(\tilde{A}_2, \tilde{B}) \in L^{<\alpha}$, then

(1) $d(\tilde{A}_1 \cup \tilde{A}_2, \tilde{B}) \ge d(\tilde{A}_1, \tilde{B}) \vee d(\tilde{A}_2, \tilde{B})$;

(2) $d(\tilde{A}_1 \cap \tilde{A}_2, \tilde{B}) = d(\tilde{A}_1, \tilde{B}) \wedge d(\tilde{A}_2, \tilde{B})$.

Proof. (1) $d(\tilde{A}_1 \cup \tilde{A}_2, \tilde{B})$

$$= \alpha \otimes S'(\tilde{A} - \tilde{A}_1 \cup \tilde{A}_2)$$

$$= \left(\alpha \to S(\tilde{A} - \tilde{A}_1 \cup \tilde{A}_2)\right)'$$

$$= \left(\alpha \to \vee_{g \in G_\alpha} \left(\tilde{A}(g) \to (\tilde{A}_1 \cup \tilde{A}_2)(g)\right)'\right)'$$

$$\geq \left(\alpha \to \left(\vee_{g \in G_\alpha} \left(\tilde{A}(g) \to \tilde{A}_1(g)\right)' \wedge \vee_{g \in G_\alpha} \left(\tilde{A}(g) \to \tilde{A}_2(g)\right)'\right)\right)'$$

$$= \left(\alpha \to S(\tilde{A} - \tilde{A}_1)\right)' \vee \left(\alpha \to S(\tilde{A} - \tilde{A}_2)\right)'$$

$$= \left(\alpha \otimes S'(\tilde{A} - \tilde{A}_1)\right) \vee \left(\alpha \otimes S'(\tilde{A} - \tilde{A}_2)\right)$$

$$= d(\tilde{A}_1, \tilde{B}) \vee d(\tilde{A}_2, \tilde{B}) .$$

(2) $d(\tilde{A}_1 \cap \tilde{A}_2, \tilde{B})$

$$= \alpha \otimes S'(\tilde{A} - \tilde{A}_1 \cap \tilde{A}_2)$$

$$= \left(\alpha \to S(\tilde{A} - \tilde{A}_1 \cap \tilde{A}_2)\right)'$$

$$= \left(\alpha \to \vee_{g \in G_\alpha} \left(\tilde{A}(g) \to (\tilde{A}_1 \cap \tilde{A}_2)(g)\right)'\right)'$$

$$= \left(\alpha \to \left(\vee_{g \in G_\alpha} \left(\tilde{A}(g) \to \tilde{A}_1(g)\right)' \vee \vee_{g \in G_\alpha} \left(\tilde{A}(g) \to \tilde{A}_2(g)\right)'\right)\right)'$$

$$= \left(\alpha \to S(\tilde{A} - \tilde{A}_1)\right)' \wedge \left(\alpha \to S(\tilde{A} - \tilde{A}_2)\right)'$$

$$= \left(\alpha \otimes S'(\tilde{A} - \tilde{A}_1)\right) \wedge \left(\alpha \otimes S'(\tilde{A} - \tilde{A}_2)\right)$$

$$= d(\tilde{A}_1, \tilde{B}) \wedge d(\tilde{A}_2, \tilde{B}) . \quad \square$$

The following theorems can be immediately obtained from Theorem 4.6 and Theorem 4.7.

Theorem 4.8 Let $K|_\alpha = (G_\alpha, M \cup \{d\}, L, \tilde{I}_\alpha)$ be a lattice-valued single-target decision context with decision value α . If $\forall \tilde{B}, \tilde{B}_1, \tilde{B}_2 \in L^M$, for $\tilde{A} \in L^{G_\alpha}$, “ $(\tilde{A}, \tilde{B}) \to \{d(\tilde{A}, \tilde{B}) = \alpha\}$ ” is a maximum decision rule and $d(\tilde{A}, \tilde{B}_1) \in L^{>\alpha}$, $d(\tilde{A}, \tilde{B}_2) \in L^{>\alpha}$, then

(1) $d(\tilde{A}, \tilde{B}_1 \cup \tilde{B}_2) = d(\tilde{A}, \tilde{B}_1) \vee d(\tilde{A}, \tilde{B}_2)$;

(2) $d(\tilde{A}, \tilde{B}_1 \cap \tilde{B}_2) \leq d(\tilde{A}, \tilde{B}_1) \wedge d(\tilde{A}, \tilde{B}_2)$.

Theorem 4.9 Let $K|_\alpha = (G_\alpha, M \cup \{d\}, L, \tilde{I}_\alpha)$ be a lattice-valued single-target decision context with decision value α . If $\forall \tilde{B}, \tilde{B}_1, \tilde{B}_2 \in L^M$, for $\tilde{A} \in L^{G_\alpha}$,

" $(\tilde{A},\tilde{B})\rightarrow\{d(\tilde{A},\tilde{B})=\alpha\}$ " is a maximum decision rule and $d(\tilde{A},\tilde{B}_1)\in L^{<\alpha}$, $d(\tilde{A},\tilde{B}_2)\in L^{<\alpha}$, then

(1) $d(\tilde{A},\tilde{B}_1\cup\tilde{B}_2)\geq d(\tilde{A},\tilde{B}_1)\vee d(\tilde{A},\tilde{B}_2)$;

(2) $d(\tilde{A},\tilde{B}_1\cap\tilde{B}_2)=d(\tilde{A},\tilde{B}_1)\wedge d(\tilde{A},\tilde{B}_2)$.

Theorem 4.10 Let $K|_\alpha=(G_\alpha,M\cup\{d\},L,\tilde{I}_\alpha)$ be a lattice-valued single-target decision context with decision value α, and $\mathbf{C}(K|_\alpha)$ the lattice-valued single-target decision concept lattice with decision value α, if $\forall\tilde{A}_1,\tilde{A}_2\in L^{G_\alpha}$, $\tilde{B}_1,\tilde{B}_2\in L^M$, " $(\tilde{A}_1,\tilde{B}_1)\rightarrow\{d(\tilde{A}_1,\tilde{B}_1)=\alpha\}$ " and " $(\tilde{A}_2,\tilde{B}_2)\rightarrow\{d(\tilde{A}_2,\tilde{B}_2)=\alpha\}$ " are the maximum decision rules, then

(1) $d(\tilde{A}_1\cup\tilde{A}_2,\tilde{B}_1\cup\tilde{B}_2)\in L^{>\alpha}$;

(2) $d(\tilde{A}_1\cap\tilde{A}_2,\tilde{B}_1\cap\tilde{B}_2)\in L^{<\alpha}$.

Proof. (1) By Definition 4.6, $(\tilde{A}_1,\tilde{B}_1)\in L(K|_\alpha)$ and $(\tilde{A}_2,\tilde{B}_2)\in L(K|_\alpha)$ hold for any $\tilde{A}_1,\tilde{A}_2\in L^{G_\alpha}$, $\tilde{B}_1,\tilde{B}_2\in L^M$, then $\tilde{B}_1=f(\tilde{A}_1)$, $\tilde{B}_2=f(\tilde{A}_2)$, $\left(\tilde{A}_1\cup\tilde{A}_2,f(\tilde{A}_1\cup\tilde{A}_2)\right)\in\mathbf{C}(K|_\alpha)$, i.e.,

" $\left(\tilde{A}_1\cup\tilde{A}_2,f(\tilde{A}_1\cup\tilde{A}_2)\right)\rightarrow\left\{d\left(\tilde{A}_1\cup\tilde{A}_2,f(\tilde{A}_1\cup\tilde{A}_2)\right)=\alpha\right\}$ " is a maximum decision rule. By the properties of lattice-valued fuzzy concept lattice,

$$\begin{aligned}d(\tilde{A}_1\cup\tilde{A}_2,\tilde{B}_1\cup\tilde{B}_2)&=d\left(\tilde{A}_1\cup\tilde{A}_2,f(\tilde{A}_1)\cup f(\tilde{A}_2)\right)\\&=d\left(\tilde{A}_1\cup\tilde{A}_2,f(\tilde{A}_1)\right)\vee d\left(\tilde{A}_1\cup\tilde{A}_2,f(\tilde{A}_2)\right)\\&\geq\alpha.\end{aligned}$$

So $d(\tilde{A}_1\cup\tilde{A}_2,\tilde{B}_1\cup\tilde{B}_2)\in L^{>\alpha}$.

(2) Similarly, $\left(\tilde{A}_1\cap\tilde{A}_2,f(\tilde{A}_1\cap\tilde{A}_2)\right)\in\mathbf{C}(K|_\alpha)$ holds for any $\tilde{A}_1,\tilde{A}_2\in L^{G_\alpha}$, $\tilde{B}_1,\tilde{B}_2\in L^M$, " $\left(\tilde{A}_1\cap\tilde{A}_2,f(\tilde{A}_1\cap\tilde{A}_2)\right)\rightarrow\left\{d\left(\tilde{A}_1\cap\tilde{A}_2,f(\tilde{A}_1\cap\tilde{A}_2)\right)=\alpha\right\}$ " is a maximum decision rule, then

$$\begin{aligned}d(\tilde{A}_1\cap\tilde{A}_2,\tilde{B}_1\cap\tilde{B}_2)&=d\left(\tilde{A}_1\cap\tilde{A}_2,f(\tilde{A}_1)\cap f(\tilde{A}_2)\right)\\&=d\left(\tilde{A}_1\cap\tilde{A}_2,f(\tilde{A}_1)\right)\wedge d\left(\tilde{A}_1\cap\tilde{A}_2,f(\tilde{A}_2)\right)\\&\leq\alpha.\end{aligned}$$

So $d(\tilde{A}_1\cap\tilde{A}_2,\tilde{B}_1\cap\tilde{B}_2)\in L^{<\alpha}$. □

Theorem 4.11 Let $K|_\alpha=(G_\alpha,M\cup\{d\},L,\tilde{I}_\alpha)$ be a lattice-valued single-target decision context with decision value α. If $\forall\tilde{A},\tilde{A}_1,\tilde{A}_2\in L^{G_\alpha}$, $\exists\tilde{B}\in L^M$, s.t., " $(\tilde{A},\tilde{B})\rightarrow\{d(\tilde{A},\tilde{B})=\alpha\}$ " is a maximum decision rule, $d(\tilde{A}_1,\tilde{B})\in L$ and $d(\tilde{A}_2,\tilde{B})\in L^{>\alpha}$, then $d(\tilde{A}_1\rightarrow\tilde{A}_2,\tilde{B})\in L^{>\alpha}$.

Proof. Suppose that (f_α, g_α) is the Galois connection on $K|_\alpha$. $\tilde{A} = h_\alpha(\tilde{B}) \subset \tilde{A}_2$ holds for any $\tilde{A}, \tilde{A}_1, \tilde{A}_2 \in L^{G_\alpha}$ and $\tilde{B} \in L^M$ by Definition 4.7, then $\tilde{A} \subset \tilde{A}_1 \to \tilde{A}_2$ holds by $\tilde{A} \to (\tilde{A}_1 \to \tilde{A}_2) = \tilde{A}_1 \to (\tilde{A} \to \tilde{A}_2) = I$, so $d(\tilde{A}_1 \to \tilde{A}_2, \tilde{B}) > d(\tilde{A}, \tilde{B})$, i.e., $d(\tilde{A}_1 \to \tilde{A}_2, \tilde{B}) \in L^{>\alpha}$. □

Theorem 4.12 Let $K|_\alpha = (G_\alpha, M \cup \{d\}, L, \tilde{I}_\alpha)$ be a lattice-valued single-target decision context with decision value α. If $\forall \tilde{B}, \tilde{B}_1, \tilde{B}_2 \in L^M$, $\exists \tilde{A} \in L^{G_\alpha}$, s.t., " $(\tilde{A}, \tilde{B}) \to \{d(\tilde{A}, \tilde{B}) = \alpha\}$ " is a maximum decision rule, $d(\tilde{A}, \tilde{B}_1) \in L$ and $d(\tilde{A}, \tilde{B}_2) \in L^{>\alpha}$, then $d(\tilde{A}, \tilde{B}_1 \to \tilde{B}_2) \in L^{>\alpha}$.

Proof. Suppose that (f_α, g_α) is the Galois connection on $K|_\alpha$. $\tilde{B} = f_\alpha(\tilde{A}) \subset \tilde{B}_2$ holds for any $\tilde{B}, \tilde{B}_1, \tilde{B}_2 \in L^M$, $\tilde{A} \in L^{G_\alpha}$ by Definition 4.7, then $\tilde{B} \subset \tilde{B}_1 \to \tilde{B}_2$ holds by $\tilde{B} \to (\tilde{B}_1 \to \tilde{B}_2) = \tilde{B}_1 \to (\tilde{B} \to \tilde{B}_2) = I$, so $d(\tilde{A}, \tilde{B}_1 \to \tilde{B}_2) > d(\tilde{A}, \tilde{B})$, i.e., $d(\tilde{A}, \tilde{B}_1 \to \tilde{B}_2) \in L^{>\alpha}$. □

4.3 Extracting Algorithm of Maximum Decision Rules

Let $K|_\alpha = (G_\alpha, M \cup \{d\}, L, \tilde{I}_\alpha)$ be a lattice-valued single-target decision context with decision value α, where $G_\alpha = \{g_1, g_2, \cdots, g_r\}$, $M = \{m_1, m_2, \cdots, m_s\}$, $L = (L_n, \vee, \wedge, ', \to, O, I)$ is a lattice implication algebra, $\tilde{I}_\alpha : G_\alpha \times M \to L$, and (f_α, h_α) is the Galois connection.

The extracting algorithm of maximum decision rules as follows:

Input: the lattice-valued single-target decision context $K|_\alpha = (G_\alpha, M \cup \{d\}, L, \tilde{I}_\alpha)$

Output: the maximum decision rules

Begin

while ($K|_\alpha \neq \Phi$) do

Calculate the fuzzy subsets $\tilde{A}_k = \left(\tilde{A}_k(g_1), \tilde{A}_k(g_2), \cdots, \tilde{A}_k(g_r)\right)$ of L^{G_α}, $(O, O, \cdots, O) \le \tilde{A}_k \le (I, I, \cdots, I)$, $1 \le k \le n^r$

For $k \leftarrow 1$ **to** n^r **do**

  for $j \leftarrow 1$ **to** s **do**

    for $i \leftarrow 1$ **to** r **do**

      $f_\alpha(g_i, m_j) := \tilde{A}_k(g_i) \to \tilde{I}_\alpha(g_i, m_j)$

      $\tilde{B}(m_j) := \tilde{B}(m_j) \wedge f_\alpha(g_i, m_j)$

    endfor;

  endfor;

  for $i \leftarrow 1$ **to** r **do**

for $j \leftarrow 1$ **to** s **do**

$h_\alpha\left(\tilde{B}(m_j)\right) := \tilde{B}(m_j) \rightarrow \tilde{I}_\alpha(g_i, m_j)$

$\tilde{C}(g_i) := \tilde{C}(g_i) \wedge h_\alpha\left(\tilde{B}(m_j)\right)$

endfor;

if $\tilde{C}(g_i) = \tilde{A}_k(g_i)$ **then**

endif;

endfor;

endfor;

end;

This above algorithm for extracting the maximum decision rules depends on the lattice-valued single-target decision concept lattice, and the decision rules basis are different with the changeable decision values of single-target decision concept lattices which provides us a direct and convenient mathematical tool.

5 Conclusions

For dealing with uncertainty problems in decision making, this paper proposed a new decision model and researched the methods of mining decision rules. Based on the lattice-valued logic, we utilized the lattice implication algebra to depict the uncertainty information and constructed the lattice-valued single-target decision concept lattice as the decision model. Concretely, we gave the definition of decision rule and the calculating method of decision rules with different decision values. We discussed the properties of decision rules and proposed the algorithm for mining the maximum decision rules, which provides a feasible method for decision making.

In the future work, we will be devoted to software implementation of this decision method and apply it into practice. Obviously, other decision models and decision methods for dealing with uncertainty information will be also our next researches.

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