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The Tourist in the Shopping Arcade[1]

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Abstract: A tourist is searching for a gift and moves along a shopping arcade until the desired object gets into sight. The location of the corresponding shop is not known in advance. Therefore in this on-line setting the tourist has to make a detour in comparison to an optimal off-line straight line path to the desired object. We can show that there is a strategy for the tourist, so that the path length is never greater than C^* times the optimal off-line path length, where $C^* = 1.059401\ldots$ holds. Furthermore, there is no strategy that attains a *competitive factor* smaller than C^*.
Key Words: Computational geometry, on-line algorithms, on-line navigation.
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1 Introduction

He, who travels, often feels the need to buy some souvenirs, to prove the greatness of the sites he saw to those who could not join him on his trip. While, generally,

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shopping is a complicated task that calls for optimizing quite conflicting goals, the average tourist wants to minimize the time it takes to find a reasonable gift.

In this paper we provide an optimal solution for the following situation. A tourist finds himself near a shopping arcade where shops are lined up in a single column; see Figure 1.

Due to reflective window panes, the tourist can see the goods on display in a shop only when he looks into its window perpendicularly. However, his eyesight is good enough to do so from the initial distance given. The tourist is prepared to scan the arcade, shop by shop, until he sees the first item that matches his criteria (highlights the site, fits into suitcase, doesn't cost too much, might please the receiver, etc.). As soon as such an item becomes visible, the tourist hurries straight into the shop to buy it. We assume that the variety of goods on offer in this arcade is such that at least one shop carries a qualifying item, so that the tourist's quest will be successful. But how should he proceed to be as quick as possible? This problem belongs to the class of on-line navigation tasks, which have attracted a lot of attention during the last 30 years; see [Borodin and El-Yaniv 1998] or [Fiat and Woeginger 1998] for surveys. Typically, one wants to accomplish a certain mission in a geometric environment that is initially not completely known. The mission may be to fully explore the environment and return to the start, or to search the environment for a goal

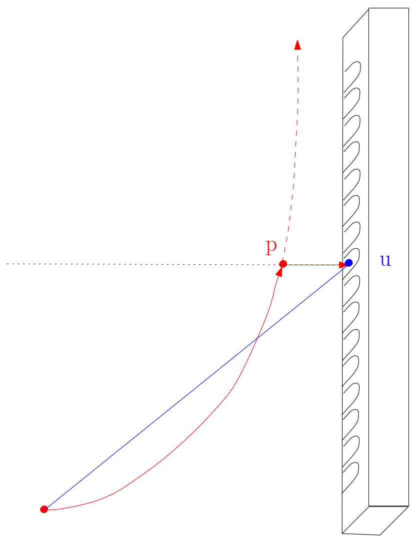

Figure 1: *The tourist in the shopping arcade.*

and stay there; the latter problem is interesting even when the environment is known but the position of the goal is unknown, see [Fleischer et al. 2008].

Equipped with complete information, one could compute an optimal solution, like the shortest round trip from which each point of the environment is visible at least once, or the shortest path from start to goal. Without complete knowledge, such an optimal solution is in general not available. But one can strive for near-optimal results.

A strategy S for solving an on-line problem P is called *C-competitive* if, for each instance I of P, the cost of solving I by means of strategy S is at most C times the cost of an optimal solution of I, plus an additive constant. Such a number C is called *competitive ratio* of strategy S for problem P. Given a problem P, the challenge is in designing a C-competitive strategy for solving it, where C is as small as possible (clearly, C cannot be smaller than 1, and it may be infinite if problem P does not admit a competitive solution at all). The concept of competitive analysis goes back to [Sleator and Tarjan 1985].

Not always are the smallest possible competitive ratios precisely known. For example, [Hagius et al. 2004] proved that no strategy can explore a simple polygon with a competitive ratio less than 1.2825; but the best exploration strategy known [Hoffmann et al. 2001] has a competitive ratio of 26.5, leaving open quite some gap. [Icking et al. 1993] considered a subproblem of polygon exploration, namely looking around a single protruding corner, and presented an optimal 1.21218-competitive strategy.

More tight results are known for the search problem. Some authors [Gal 1980, Baeza-Yates et al. 1993, Kao et al. 1996] studied the problem of finding a point on a line. Exploring both sides in turn, each time doubling the depth from the start, results in a competitive ratio of 9, and no smaller ratio can be achieved by any deterministic strategy. This result can be generalized to $m \geq 2$ halflines that meet at the start; see the work in [López-Ortiz and Schuierer 2001] and [López-Ortiz and Schuierer 2004]. An optimal strategy for searching a point in a "street" polygon was given in [Icking et al. 2004]. The old problem of searching for a point in the plane was recently settled by [Langetepe 2010].

In our tourist's case, a problem instance is given by the start point $s = (0, 0)$, by the vertical half-line H running from $(1, 0)$ upwards, and by some point $u = (1, h)$ on H representing the bottommost shop that carries a suitable item; see Figure 2.[2]

If the location of u were known to the tourist, he would walk straight from s to u, at an optimum cost of $|su| = \sqrt{1 + h^2}$. Since u is not known, the tourist needs to follow a path S—representing the strategy—that leads him upwards, so that he eventually reaches the horizontal line $\{Y = h\}$ at some point $p = (w(h), h)$

[2] We may assume that the distance from startpoint s to halfline H equals 1 since our problem is invariant under scaling.

to the left of, or on, half-line H. From here he is able to see u and can walk horizontally from p to u. Let $L(h)$ denote the length of S between s and p. Then the total length of the tourist's path is given by $L(h) + 1 - w(h)$; see Figure 2. Hence, the competitive ratio of strategy S equals

$$C(S) \;=\; \max_{h \geq 0} \frac{L(h) + 1 - w(h)}{\sqrt{1 + h^2}}. \tag{1}$$

The path leading vertically upwards would have a competitive ratio of $\sqrt{2} \approx$

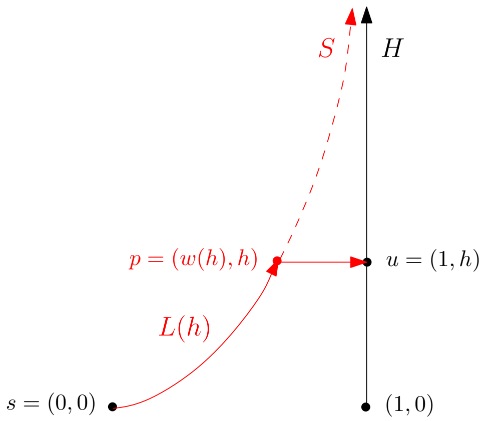

Figure 2: *Notations used.*

1.4142; it is attained by $u = (1, 1)$. The same holds for the path that first runs horizontally to $(1, 0)$, and then upwards along H. In order to improve on this ratio, the tourist should move upwards and to the right simultaneously. A suitable path S^* will be presented in Section 2. We shall derive some structural properties of S^*, and the fact that $C(S^*) = 1.059401\ldots$ holds. Then, in Section 3, we show that no path can attain a competitive ratio smaller than $C(S^*)$, thus proving S^* to be an optimum solution to the tourist's problem.

2 Constructing a Strategy

Apparently any reasonable strategy should move simultaneously along and towards the arcade; that is, in positive X- and Y-direction. Note that the competitive factor for any reasonable strategy converges to 1 for goals with very small Y-coordinate and also for goals with a large Y-coordinate.

Therefore, for the first part of our strategy will make use of a line segment and the competitive ratio will increase for a while. The second part will be a curve that converges toward the arcade. As in many similar application the ratio for this part of the strategy will be the same for all corresponding goals on H. After hitting the arcade, the third part of the strategy just moves along the arcade and the competitive ratio will therefore decrease.

Altogether, we are presenting a path S^* that consists of three parts, a line segment connecting startpoint $s = (0, 0)$ to some point (a, b), followed by a curve that hits halfline H in some point $(1, d)$. From there on, S^* runs along H; see Figure 3. The points on path S^* will be parametrized by their X-coordinates $w(h)$ as a function of their Y-coordinates, h. The parameters a, b, d and the

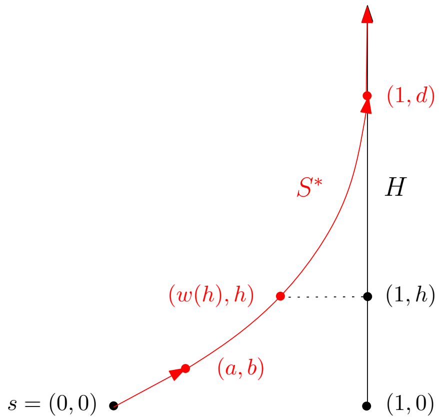

Figure 3: *Defining strategy S^*.*

function $w(h)$ are specified as follows. Using the standard abbreviation

$$\operatorname{arctanh}(x) = \frac{1}{2} \ln \frac{1+x}{1-x}, \text{ where } x \in (-1, 1),$$

we can define a function

$$f(y, h) \;:=\; \frac{\operatorname{arctanh}\left(\frac{1}{\sqrt{1+y^2}}\right) + \sqrt{1+y^2} - \operatorname{arctanh}\left(\frac{1}{\sqrt{1+h^2}}\right) - y^2\sqrt{1+h^2}}{2\sqrt{1+y^2}} \tag{2}$$

for all non-zero values of y and h. Let

$$g(y) := f\left(y, \frac{1}{y}\right).$$

Since $g(0.1) = 1.892\ldots$ and $g(1) = 0$, there must be a number $b \in (0.1, 1)$ such that $g(b) = 1$ holds, by continuity. Moreover, b is uniquely determined, because function g has a negative derivative in this interval. Numerical computation shows that $b = 0.349748\ldots$ holds. We fix this number b and let

$$a := f(b, b) = \frac{1 - b^2}{2} = 0.438835\ldots \quad \text{and} \quad d := \frac{1}{b} = 2.85912\ldots$$

Now we are ready to define the function $w(h)$ as follows.

$$w(h) := \begin{cases} \frac{a}{b} h, & 0 \le h \le b \\ f(b, h), & b \le h \le d \\ 1, & b \ge d \end{cases} \tag{3}$$

By definition, $w(h)$ is continuous and fulfills $w(0) = 0$, $w(b) = a$, and $w(h) = 1$ for $h \ge d$. The resulting path $S^*(h) = (w(h), h)$ has some extremal properties that are stated in the following Lemma 1. Let

$$C(h) := \frac{L(h) + 1 - w(h)}{\sqrt{1 + h^2}} \tag{4}$$

denote the ratio in path length that results if the target u is located at height h; see Formula (1).

Lemma 1. *Function $C(h)$ is*

1. *monotonically increasing from 1 for $h \in [0, b]$, attaining a maximum of $C(b) = \sqrt{1 + b^2} =: C^* = 1.059401\ldots$;*
2. *constantly equal to C^* for $h \in [b, d]$;*
3. *monotonically decreasing towards 1 for $h \ge d$.*

Proof. For h between 0 and b we have $w(h) = ha/b$, so that $L(h) = \sqrt{(\frac{ha}{b})^2 + h^2}$ holds for the length of the line segment from $(0, 0)$ to $(w(h), h)$. This yields for the derivative of $C(h)$

$$\begin{aligned} C'(h) &= \frac{(L'(h) - w'(h))\sqrt{1 + h^2} - (L(h) + 1 - w(h))\,\frac{1}{2}\frac{1}{\sqrt{1+h^2}} 2h}{1 + h^2} \\ &= \frac{(L'(h) - w'(h))\,(1 + h^2) - (L(h) + 1 - w(h))\,h}{(1 + h^2)^{3/2}}. \end{aligned}$$

Numerator N of this expression equals

$$\begin{aligned} N &= \left(\sqrt{\frac{a^2}{b^2} + 1} - \frac{a}{b}\right)(1 + h^2) - \left(h\sqrt{\frac{a^2}{b^2} + 1} + 1 - \frac{ha}{b}\right) h \\ &= \sqrt{\frac{a^2}{b^2} + 1} - \left(\frac{a}{b} + h\right) \\ &\ge \sqrt{\frac{a^2}{b^2} + 1} - \left(\frac{a}{b} + b\right) = 0 \end{aligned}$$

since $a = (1 - b^2)/2$ implies $\frac{a^2}{b^2} + 1 = (\frac{a}{b} + b)^2$. This shows that $C(h)$ is monotonically increasing. Clearly, $C(0) = 1$ holds, and

$$\begin{aligned} C(b) &= \frac{L(b) + 1 - w(b)}{\sqrt{1 + b^2}} \\ &= \frac{\sqrt{a^2 + b^2} + 1 - a}{\sqrt{1 + b^2}} \\ &= \sqrt{1 + b^2} = C^* = 1.059401\ldots \end{aligned}$$

follows from $\sqrt{a^2 + b^2} = b^2 + a$. This proves the first claim of our lemma.

Now let us assume that $h \in [b, d]$. Using $\operatorname{arctanh}'(x) = 1/(1 - x^2)$ we obtain from Formulae (3) and (2), by straightforward calculation of the derivative,

$$w'(h) \;=\; \frac{1}{2C^*} \, \frac{1 - b^2 h^2}{h\sqrt{1 + h^2}}, \tag{5}$$

which implies

$$2C^* w'(h) \frac{h}{\sqrt{1 + h^2}} \;=\; 1 - \frac{{C^*}^2 h^2}{1 + h^2},$$

hence

$$1 + w'(h)^2 = w'(h)^2 + 2w'(h) C^* \frac{h}{\sqrt{1 + h^2}} + \frac{{C^*}^2 h^2}{1 + h^2}.$$

Since $w'(h) \geq 0$, thanks to $h \leq d = 1/b$, we can conclude that

$$\sqrt{1 + w'(h)^2} = w'(h) + C^* \frac{h}{\sqrt{1 + h^2}}$$

or

$$-w'(h) + \sqrt{1 + w'(h)^2} = C^* \frac{h}{\sqrt{1 + h^2}}$$

holds. By integration, we obtain

$$A - w(h) + \int_b^h \sqrt{1 + w'(t)^2} \, dt = C^* \sqrt{1 + h^2}$$

for some constant A. For $h = b$, $w(b) = a$ holds, and therefore $A = 1 + a + b^2 = 1 + \sqrt{a^2 + b^2}$ results. But

$$\sqrt{a^2 + b^2} + \int_b^h \sqrt{1 + w'(t)^2} \, dt + 1 - w(h) = C^* \sqrt{1 + h^2}$$

means $C(h) = C^*$, thus proving the second claim of the lemma; compare Formula (4).

Finally, let us consider the behavior of the function $C(h)$ for $h = d+y$, where $y \geq 0$. Let L denote the length of S^* between $(0,0)$ and $(1,d)$. Then,

$$C(d+y) = \frac{L+y}{\sqrt{1+(d+y)^2}}.$$

The derivative with respect to y equals

$$\frac{d}{dy}C(d+y) = \frac{1+(d+y)^2-(L+y)(d+y)}{(1+(d+y)^2)^{3/2}}.$$

For the numerator M we find

$$M \leq 0 \;\Leftrightarrow\; 1+d^2-Ld \leq y(L-d).$$

The latter assertion is true since $L > d$ holds, and because

$$C(d) = \frac{L}{\sqrt{1+d^2}} = C^* = \sqrt{1+b^2}$$

and $b = 1/d$ imply $Ld = d^2+1$. This completes the proof of Lemma 1.

We note that the definition of the function $w(h)$ can be found by working through the proof of Lemma 1 backwards.

3 Proof of Optimality

In this section we show that S^* is an optimum solution for the tourist's problem. The following fact will be helpful.

Lemma 2. *Path S^* is convex.*

Proof. From Formula (5) we obtain

$$w''(h) = -\frac{b^2h^2+2h^2+1}{2(1+h^2)^{3/2}\sqrt{1+b^2h^2}} \leq 0$$

for all arguments $h \in [b,d]$. Moreover, the line segment from $(0,0)$ to (a,b)— the initial part of S^*—has the same orientation as the tangent to S^* at (a,b); this follows from

$$w'(b) = \frac{1}{2C^*}\,\frac{1-b^4}{b\sqrt{1+b^2}} = \frac{1-b^4}{2b(1+b^2)} = \frac{1-b^2}{2b} = \frac{a}{b}.$$

Now we can prove our main result.

Theorem 3. *Strategy S^* constructed in Section 2 has a competitive ration of* $1.059401\ldots$. *No other strategy attains a ratio smaller than this.*

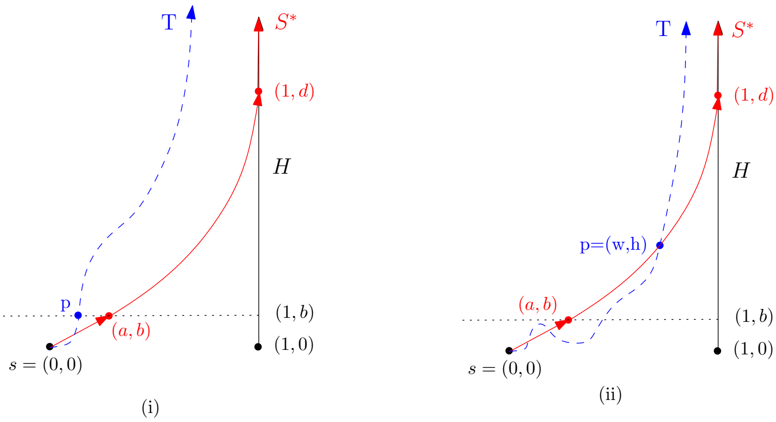

Figure 4: *Proving optimality of strategy S^*.*

Proof. Lemma 1 immediately implies

$$C(S^*) = C^* = 1.059401\ldots$$

Let T denote an arbitrary strategy, represented by a path that starts from $s = (0, 0)$ and runs upwards, to the left of, or on, halfline H. First, let us assume that T passes through a point of height b to the left of (a, b), and let p be the first point on T of this type; see Figure 4(i). Then the length of T between s and p, plus the distance $|p(1, b)|$, exceeds $L(b) + 1 - a$. Thus, $C(T) > C(b) = C^*$, by Lemma 1.

If this case does not apply, T runs between S^* and H. Suppose that T meets S^* at some point p, that might be equal to $(1, d)$; see Figure 4(ii). If the part of T from s to p is not the same as the corresponding part of S^*, it must be strictly longer, by the convexity property established in Lemma 2. Then, if p is at height h, we have $C(T) > C(h) = C^*$, by Lemma 1.

This shows that $C(T) > C^*$ holds, unless T coincides with S^* between s and $(1, d)$, in which case $C(T) = C^*$ trivially follows.

4 Conclusion

In this paper we have shown how to optimally move to a point on a half-line, starting from a point that is a certain distance away. An obvious question is, if this result can be combined with classical 2-way search on two halflines. Another natural generalization would be the following. Here we have shown how to reach the origin of a ray that emanates at a right angle from a given half-line (the arcade), given that the ray can only be detected by stepping on it, just as for the beam of a tightly focussed flashlight. How about detecting general rays in the plane? Perhaps some variation of spiral search will solve this problem optimally; compare [Eubeler et al. 2006].

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References

[Baeza-Yates et al. 1993] R. Baeza-Yates, J. Culberson, and G. Rawlins. Searching in the plane. *Inform. Comput.*, 106:234–252, 1993.

[Borodin and El-Yaniv 1998] A. Borodin and R. El-Yaniv. *Online Computation and Competitive Analysis*. Cambridge University Press, 1998.

[Eubeler et al. 2006] A. Eubeler, R. Fleischer, T. Kamphans, R. Klein, E. Langetepe, and G. Trippen. Competitive online searching for a ray in the plane. In S. Fekete, R. Fleischer, R. Klein, and A. López-Ortiz, editors, *Robot Navigation*, number 06421 in Dagstuhl Seminar Proceedings. Internationales Begegnungs- und Forschungszentrum für Informatik (IBFI), Schloss Dagstuhl, Germany, 2006.

[Fiat and Woeginger 1998] A. Fiat and G. Woeginger, editors. *On-line Algorithms: The State of the Art*, volume 1442 of *Lecture Notes Comput. Sci.* Springer-Verlag, 1998.

[Fleischer et al. 2008] R. Fleischer, T. Kamphans, R. Klein, E. Langetepe, and G. Trippen. Competitive online approximation of the optimal search ratio. *Siam J. Comput.*, pages 881–898, 2008.

[Gal 1980] S. Gal. *Search Games*, volume 149 of *Mathematics in Science and Engineering*. Academic Press, New York, 1980.

[Hagius et al. 2004] R. Hagius, C. Icking, and E. Langetepe. Lower bounds for the polygon exploration problem. In *Abstracts 20th European Workshop Comput. Geom.*, pages 135–138. Universidad de Sevilla, 2004.

[Hoffmann et al. 2001] F. Hoffmann, C. Icking, R. Klein, and K. Kriegel. The polygon exploration problem. *SIAM J. Comput.*, 31:577–600, 2001.

[Icking et al. 2004] C. Icking, R. Klein, E. Langetepe, S. Schuierer, and I. Semrau. An optimal competitive strategy for walking in streets. *SIAM J. Comput.*, 33:462–486, 2004.

[Icking et al. 1993] C. Icking, R. Klein, and L. Ma. The optimal way for looking around a corner. In *Proc. 4th IEEE–IEE Vehicle Navigation and Informations Systems Conference*, 1993.

[Kao et al. 1996] M.-Y. Kao, J. H. Reif, and S. R. Tate. Searching in an unknown environment: An optimal randomized algorithm for the cow-path problem. *Inform. Comput.*, 133(1):63–79, 1996.

[Langetepe 2010] E. Langetepe. On the optimality of spiral search. In *SODA 2010: Proc. 21st Annu. ACM-SIAM Symp. Disc. Algor.*, pages 1–12, 2010.

[López-Ortiz and Schuierer 2001] A. López-Ortiz and S. Schuierer. The ultimate strategy to search on m rays? *Theor. Comput. Sci.*, 261(2):267–295, 2001.

[López-Ortiz and Schuierer 2004] A. López-Ortiz and S. Schuierer. Online parallel heuristics, processor scheduling, and robot searching under the competitive framework. *Theoret. Comput. Sci.*, 310:527–537, 2004.

[Sleator and Tarjan 1985] D. D. Sleator and R. E. Tarjan. Amortized efficiency of list update and paging rules. *Commun. ACM*, 28:202–208, 1985.