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# Computable Separation in Topology, from $T_0$ to $T_2$

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**Abstract:** This article continues the study of computable elementary topology started in [Weihrauch and Grubba 2009]. For computable topological spaces we introduce a number of computable versions of the topological separation axioms $T_0$, $T_1$ and $T_2$. The axioms form an implication chain with many equivalences. By counterexamples we show that most of the remaining implications are proper. In particular, it turns out that computable $T_1$ is equivalent to computable $T_2$ and that for spaces without isolated points the hierarchy collapses, that is, the weakest computable $T_0$ axiom $\mathrm{WCT}_0$ is equivalent to the strongest computable $T_2$ axiom $\mathrm{SCT}_2$. The $SCT_2$-spaces are closed under Cartesian product, this is not true for most of the other classes of spaces. Finally we show that the computable version of a basic axiom for an effective topology in intuitionistic topology is equivalent to $\mathrm{SCT}_2$.

**Key Words:** computable analysis, computable topology, axioms of separation
**Category:** F.0, F.m, G.0, G.m

## 1 Introduction

This article continues with the study of computable topology started in [Weihrauch and Grubba 2009]. For computable topological spaces (as defined in [Weihrauch and Grubba 2009]) we define a number of computable versions of the topological $T_0$-, $T_1$- and $T_2$-axioms and study their relation. We will use the notation and results from [Weihrauch and Grubba 2009] some of which are mentioned very shortly in Section 2.

In Section 3 we introduce a number of axioms for computable separation of points in $T_0$-, $T_1$- and $T_2$-spaces. We show that the axioms are logically equivalent for equivalent computable topological spaces, where two computable topological spaces are equivalent, if they induce the same computability on the points [Weihrauch and Grubba 2009, Definition 21].

In Section 4 we prove all the implications between these axioms. They form a linear hierarchy with several equivalences. Surprisingly, computable $T_1$ and computable $T_2$ are equivalent. The hierarchy collapses for spaces with no singleton open sets. We characterize the strongest axiom $\mathrm{SCT}_2$ and give a sufficient condition for it.

In Section 5 we give counterexamples for all the implications introduced in Section 4 that are proper.

For $T_2$-spaces also compact sets can be separated by open neighborhoods. In Section 6 we define some computable versions of separating compact sets and study their relation. We also introduce the computable version IT of a basic separation axiom from intuitionistic topology and prove IT $\iff$ $\mathrm{SCT}_2$.

If some of the introduced axioms holds for a computable topological space then it also holds for every subspace. The strongly computable $T_2$-spaces are closed under Cartesian product, this is not true for most of the other axioms. This is shown in Section 7.

Some computable separation axioms have been used in [Schröder 1998, Grubba et al. 2007, Grubba et al. 2007, Xu and Grubba 2009], where, however, for a computable topological space the basis sets must be non-empty. Some results of this article have already been proved (as weaker versions) in [Grubba et al. 2007, Xu and Grubba 2009].

## 2 Preliminaries

We will use the terminology and abbreviations summarized in [Weihrauch and Grubba 2009, Section 2] and also results from [Weihrauch and Grubba 2009]. For further details see [Weihrauch 2000, Weihrauch 2008, Brattka et al. 2008].

By $\Sigma$ we denote a sufficiently large finite alphabet such that $0, 1 \in \Sigma$. As usual, $\Sigma^*$ is the set of finite words and $\Sigma^\omega$ is the set of infinite sequences of symbols from $\Sigma$. Let $\Sigma$ be a finite alphabet such that $0, 1 \in \Sigma$. By $\Sigma^*$ we denote the set of finite words over $\Sigma$ and by $\Sigma^\omega$ the set of infinite sequences $p : \mathbb{N} \to \Sigma$ over $\Sigma$, $p = (p(0)p(1)\ldots)$. For a word $w \in \Sigma^*$ let $|w|$ be its length and let $\varepsilon \in \Sigma^*$ be the empty word. For $p \in \Sigma^\omega$ let $p^{<i} \in \Sigma^*$ be the prefix of $p$ of length $i \in \mathbb{N}$. We use the "wrapping function" $\iota : \Sigma^* \to \Sigma^*$, $\iota(a_1a_2\ldots a_k) := 110a_10a_20\ldots a_k011$ for coding words such that $\iota(u)$ and $\iota(v)$ cannot overlap properly. Let $\langle i, j\rangle := (i+j)(i+j+1)/2+j$ be the bijective Cantor pairing function on $\mathbb{N}$. We consider standard functions for finite or countable tupling on $\Sigma^*$ and $\Sigma^\omega$ denoted by $\langle\cdot\rangle$ [Weihrauch 2000, Definition 2.1.7], in particular, $\langle u_1, \ldots, u_n\rangle = \iota(u_1)\ldots\iota(u_n)$, $\langle u, p\rangle = \iota(u)p$, $\langle p, q\rangle = (p(0)q(0)p(1)q(1)\ldots)$ and $\langle p_0, p_1, \ldots\rangle\langle i, j\rangle = p_i(j)$ for $u, u_1, u_2, \ldots \in \Sigma^*$ and $p, q, p_0, p_1, \ldots \in \Sigma^\omega$. For $u \in \Sigma^*$ and $w \in \Sigma^* \cup \Sigma^\omega$ let $u \ll w$ iff $\iota(u)$ is a subword of $w$ and let $\widehat{w}$ be the longest subword $v \in 11\Sigma^*11$ of $w$ (and the empty word if no such subword exists). Then for $u, w_1, w_2 \in \Sigma^*$, $(u \ll w_1 \vee u \ll w_2) \iff u \ll \widehat{w}_1\widehat{w}_2$.

For $Y_0, \ldots, Y_n \in \{\Sigma^*, \Sigma^\omega\}$ a partial function $f :\subseteq Y_1 \times \ldots \times Y_n \to Y_0$ is computable, if it is computed by a Type-2 machine. A Type-2 machine $M$ is a Turing machine withe $n$ input tapes, one output tape and finitely many additional work tapes. A specification assigns to the input tapes $1, \ldots, n$ and the output tape 0 types $Y_i \in \{\Sigma^*, \Sigma^\omega\}$ such that the machine computes a function $f_M :\subseteq Y_1 \times \ldots \times Y_n \to Y_0$ [Weihrauch 2000]. Notice that on the output tape the machine can only write and move its head to the right.

A notation of a set $X$ is a surjective partial function $\nu :\subseteq \Sigma^* \to X$ and a representation is a surjective partial function $\delta :\subseteq \Sigma^\omega \to X$. Here, finite or infinite sequences of symbols are considered as "concrete names" of the "abstract" elements of $X$. Computability on $X$ is defined by computations on names. Let $\gamma_i :\subseteq Y_i \to X_i$, $Y_i \in \{\Sigma^*, \Sigma^\omega\}$ for $i \in \{0,1\}$ be notations or representations. A set $W \subseteq X_0$ is called $\gamma_0$-r.e. (recursively enumerable), if there is a Type-2 machine $M$ that halts on input $y_0 \in \mathrm{dom}(\gamma_0)$ iff $\gamma_0(y_0) \in W$. A function $h :\subseteq Y_1 \to Y_0$ realizes a multi-function $f : X_1 \rightrightarrows X_0$, iff $\gamma_0 \circ h(y_1) \in f \circ \gamma_1(y_1)$ whenever $f \circ \gamma_1(y_1)) \neq \emptyset$. The function $f$ is called $(\gamma_1, \gamma_0)$-computable, if it has a computable realization. The definitions can be generalized straightforwardly to subsets of $X_1 \times \ldots \times X_n$ and multi-functions $f : X_1 \times \ldots \times X_n \to X_0$ ($(\gamma_1, \ldots, \gamma_n)$-r.e., $(\gamma_1, \ldots, \gamma_n, \gamma_0)$-computable).

In this article we study axioms of computable separation for *computable topological spacees* $\mathbf{X} = (X, \tau, \beta, \nu)$ [Weihrauch and Grubba 2009, Definition 4], where $\tau$ is a $T_0$-topology on the set $X$ and $\nu :\subseteq \Sigma^* \to \beta$ is a notation of a base $\beta$ of $\tau$ such that $\mathrm{dom}(\nu)$ is recursive and there is an r.e. set $S \subseteq (\mathrm{dom}(\nu))^3$ such that $\nu(u) \cap \nu(v) = \bigcup\{\nu(w) \mid (u, v, w) \in S\}$. We mention expressly that in the past various spaces have been called "computable topological space". We allow $U = \emptyset$ for $U \in \beta$ which is forbidden in, for example, [Grubba et al. 2007, Xu and Grubba 2009].

We define a notation $\nu^{\mathrm{fs}}$ of the finite subsets of the base $\beta$ by $\nu^{\mathrm{fs}}(w) = W$ $:\iff$ $((\forall v \ll w) v \in \mathrm{dom}(\nu) \ \wedge \ W = \{\nu(v) \mid v \ll w\})$. Then $\bigcup \nu^{\mathrm{fs}}$ and $\bigcap \nu^{\mathrm{fs}}$ are notations of the finite unions and the finite intersections of base elements, respectively.

For the points of $X$ we consider the canonical (or inner) representation $\delta :\subseteq \Sigma^\omega \to X$; $\delta(p) = x$ iff $p$ is a list of all $\iota(u)$ (possibly padded with 1s) such that $x \in \nu(u)$. For the set of open sets, the topology $\tau$ we consider the inner representation $\theta :\subseteq \Sigma^\omega \to \tau$ defined by $u \in \mathrm{dom}(\nu)$ if $u \ll p \in \mathrm{dom}(\theta)$ and $\delta(p) := \bigcup\{\nu(u) \mid u \ll p\}$. For the closed sets we consider the outer representation $\psi^-(p) := X \setminus \theta(p)$. Finally, for the set of compact sets (of $T_2$-spaces we consider the cover representation defined by $u \in \mathrm{dom}(\nu^{\mathrm{fs}})$ if $u \ll p \in \mathrm{dom}(\kappa)$ and $\kappa(p) = K$ iff $p$ is a list of all $\iota(u)$ such that $K \subseteq \bigcup \nu^{\mathrm{fs}}(u)$ [Weihrauch 2000, Weihrauch and Grubba 2009].

## 3 Axioms for Computable Separation of Points

For a topological space $\mathbf{X} = (X, \tau)$ we consider the following separation properties:

**Definition 1 (classical separation axioms).**

$\mathrm{T}_0 : (\forall x, y \in X,\ x \neq y)(\exists W \in \tau)((x \in W \wedge y \notin W) \vee (x \notin W \wedge y \in W)))$,
$\mathrm{T}_1 : (\forall x, y \in X,\ x \neq y)(\exists W \in \tau)(x \in W \wedge y \notin W)$,
$\mathrm{T}_2 : (\forall x, y \in X,\ x \neq y)(\exists U, V \in \tau)(U \cap V = \emptyset \wedge x \in U \wedge y \in V)$.

For $i = 0, 1, 2$, we call $\mathbf{X} = (X, \tau)$ a $T_i$-space iff $\mathrm{T_i}$ is true. Obviously, $\mathrm{T}_2 \Longrightarrow \mathrm{T}_1 \Longrightarrow \mathrm{T}_0$, where the implications are proper [Engelking 1989]. $T_2$-spaces are called *Hausdorff spaces*. We mention that $(X, \tau)$ is a $T_1$-space, iff all sets $\{x\}$ $(x \in X)$ are closed [Engelking 1989].

In this article we study computable topological spaces $\mathbf{X} = (X, \tau, \beta, \nu)$, which by definition are $T_0$-spaces with countable base (also called *second countable* $T_0$-spaces). First, we introduce computable versions $\mathrm{CT_i}$ of the conditions $\mathrm{T_i}$ by requiring that the existing open neighborhoods can be computed. For the points we compute basic neighborhoods (see Lemma 4.1 below).

**Definition 2 (axioms of computable separation).** For $i \in \{0, 1, 2\}$ define conditions $\mathrm{CT_i}$ as follows.

$\mathrm{CT}_0$ : The multi-function $t_0$ is $(\delta, \delta, \nu)$-computable where $t_0$ maps every $(x, y) \in X^2$ such that $x \neq y$ to some $U \in \beta$ such that

$$(x \in U \text{ and } y \notin U) \text{ or } (x \notin U \text{ and } y \in U). \tag{1}$$

$\mathrm{CT}_1$ : The multi-function $t_1$ is $(\delta, \delta, \nu)$-computable, where $t_1$ maps every $(x, y) \in X^2$ such that $x \neq y$ to some $U \in \beta$ such that $x \in U$ and $y \notin U$.

$\mathrm{CT}_2$ : The multi-function $t_2$ is $(\delta, \delta, [\nu, \nu])$-computable, where $t_2$ maps every $(x, y) \in X^2$ such that $x \neq y$ to some $(U, V) \in \beta^2$ such that $U \cap V = \emptyset$, $x \in U$ and $y \in V$.

Obviously, $\mathrm{CT_i}$ implies $\mathrm{T_i}$. We introduce some further computable $T_i$-conditions.

**Definition 3 (further axioms of computable separation).**
$\mathrm{WCT}_0$ : There is an r.e. set $H \subseteq \mathrm{dom}(\nu) \times \mathrm{dom}(\nu)$ such that

$$(\forall x, y,\ x \neq y)(\exists (u, v) \in H)(x \in \nu(u) \ \wedge\ y \in \nu(v)) \quad \text{and} \tag{2}$$

$$(\forall (u, v) \in H) \begin{cases} \quad \nu(u) \cap \nu(v) = \emptyset \\ \vee\ (\exists x)\, \nu(u) = \{x\} \subseteq \nu(v) \\ \vee\ (\exists y)\, \nu(v) = \{y\} \subseteq \nu(u)\,. \end{cases} \tag{3}$$

$\mathrm{SCT}_0$ : The multi-function $t_0^s$ is $(\delta, \delta, [\nu_{\mathbb{N}}, \nu])$-computable where $t_0^s$ maps every $(x, y) \in X^2$ such that $x \neq y$ to some $(k, U) \in \mathbb{N} \times \beta$ such that $(k = 1,\ x \in U \text{ and } y \notin U)$ or $(k = 2,\ x \notin U \text{ and } y \in U)$.

$\mathrm{CT}_0'$ : There is an r.e. set $H \subseteq \mathrm{dom}(\nu_{\mathbb{N}}) \times \mathrm{dom}(\nu) \times \mathrm{dom}(\nu)$ such that

$$(\forall x, y,\ x \neq y)(\exists (w,u,v) \in H)(x \in \nu(u)\ \wedge\ y \in \nu(v)) \quad \text{and} \tag{4}$$

$$(\forall (w,u,v) \in H) \begin{cases} \quad \nu(u) \cap \nu(v) = \emptyset \\ \vee\ \nu_{\mathbb{N}}(w) = 1 \wedge (\exists x)\, \nu(u) = \{x\} \subseteq \nu(v) \\ \vee\ \nu_{\mathbb{N}}(w) = 2 \wedge (\exists y)\, \nu(v) = \{y\} \subseteq \nu(u)\,. \end{cases} \tag{5}$$

$\mathrm{CT}_1'$ : There is an r.e. set $H \subseteq \Sigma^* \times \Sigma^*$ such that

$$(\forall x, y,\ x \neq y)(\exists (u,v) \in H)(x \in \nu(u) \wedge y \in \nu(v)) \quad \text{and} \tag{6}$$

$$(\forall (u,v) \in H) \begin{cases} \quad \nu(u) \cap \nu(v) = \emptyset \\ \vee\ (\exists x)\, \nu(u) = \{x\} \subseteq \nu(v)\,. \end{cases} \tag{7}$$

$\mathrm{CT}_2'$ : There is an r.e. set $H \in \Sigma^* \times \Sigma^*$ such that

$$(\forall x, y,\ x \neq y)(\exists (u,v) \in H)(x \in \nu(u) \wedge y \in \nu(v)) \quad \text{and} \tag{8}$$

$$(\forall (u,v) \in H) \begin{cases} \quad \nu(u) \cap \nu(v) = \emptyset \\ \vee\ (\exists x)\, \nu(u) = \{x\} = \nu(v)\,. \end{cases} \tag{9}$$

$\mathrm{SCT}_2$ : There is an r.e. set $H \in \Sigma^* \times \Sigma^*$ such that

$$(\forall x, y,\ x \neq y)(\exists (u,v) \in H)(x \in \nu(u) \wedge y \in \nu(v)) \quad \text{and} \tag{10}$$

$$(\forall (u,v) \in H)\, \nu(u) \cap \nu(v) = \emptyset\,. \tag{11}$$

We do not consider the numerous variants of the separation axioms where in some places the representations $\delta$ of the points, $\theta$ of the open sets and $\psi^-$ of the closed sets are replaced by $\delta^-$, $\theta^-$ and $\psi^+$, respectively [Weihrauch and Grubba 2009, Definition 5].

In contrast to $\mathrm{CT}_0$, in $\mathrm{SCT}_0$ the separating function gives immediate information about the direction of the separation. Also in $\mathrm{CT}_0'$ some information about the direction of the separation is included while no such information is given in the weak version $\mathrm{WCT}_0$. $\mathrm{CT}_0'$, $\mathrm{CT}_1'$ and $\mathrm{CT}_2'$ are versions of $\mathrm{CT}_0$, $\mathrm{CT}_1$ and $\mathrm{CT}_2$, respectively where base sets are used instead of points (see Theorem 5 below). The strong version $\mathrm{SCT}_2$ results from $\mathrm{CT}_2'$ by excluding the case $(\exists x)\, \nu(u) = \{x\} = \nu(v)$. Notice that $\mathrm{SCT}_2$ results also from $\mathrm{WCT}_0$, $\mathrm{CT}_0'$ and $\mathrm{CT}_1'$ by excluding the corresponding cases. The following examples illustrate the definitions. Further examples are given in Section 5.

*Example 1.* 1. ($SCT_2$) The *computable real line* is defined by $\mathbf{R} := (\mathbb{R}, \tau_{\mathbb{R}}, \beta, \nu)$ such that $\tau_{\mathbb{R}}$ is the real line topology and $\nu$ is a canonical notation of the set of all open intervals with rational endpoints. $\mathbf{R}$ is a computable topological space. The set $H := \{(u,v) \mid \nu(u) \cap \nu(v) = \emptyset\}$ is r.e. Therefore the computable real line is an $\mathrm{SCT}_2$-space.

2. ($T_0$ but not $WCT_0$) The computable lower real line is defined by $\mathbf{R}_< := (\mathbb{R}, \tau_<, \beta_<, \nu_<)$ where $\nu_<(w) := (\nu_{\mathbb{Q}}; \infty)$. $\mathbf{R}_<$ is $T_0$ but not $T_1$. Suppose it is $WCT_0$. Then by (3) $H = \emptyset$ since for any two base elements $U$ and $V$, $U$ is not a singleton and $U \cap V \neq \emptyset$. But $H \neq \emptyset$ by (2).

3. ($CT_0$ but not $T_1$) The Sierpinski space defined by $\mathbf{Si} := (\{\bot, \top\}, \tau_{\mathbf{Si}}, \beta_{\mathbf{Si}}, \nu_{\mathbf{Si}})$ such that $\nu_{\mathbf{Si}}(0) = \{\bot, \top\}$ and $\nu_{\mathbf{Si}}(1) = \{\top\}$. For $p, q \in \Sigma^\omega$ let $f(p,q) := 1 \in \Sigma^*$. Then $f$ realizes $t_0$ since for $x \neq y$, $x \in \nu_{\mathbf{Si}}(1) \iff y \notin \nu_{\mathbf{Si}}(1)$.

4. ($T_1$ but not $T_2$ or $WCT_0$) Let $\mathbf{X} = (\mathbb{N}, \tau, \beta, \nu)$ such that $\tau = \beta$ is the set of cofinite subsets of $\mathbb{N}$ and $\nu$ is a canonical notation of $\beta$. Then $\mathbf{X}$ is a computable topological space. $\mathbf{X}$ is $T_1$ since singletons $\{x\}$ are closed but not not $T_2$ since the intersection of any two non-empty open sets is not empty. Suppose $\mathbf{X}$ is $WCT_0$. Then by (3) $H = \emptyset$ since for any two base elements $U$ and $V$, $U$ is not a singleton and $U \cap V \neq \emptyset$. But $H \neq \emptyset$ by (2).

By the next lemma, in the above computable separation axioms the notation $\nu$ of the base can be replaced by the representation $\theta$ of the open sets and the axioms are robust, that is, they do not depend on the notation $\nu$ of the base explicitly but only on the computability concept on the points induced by it [Weihrauch and Grubba 2009, Definition 21, Theorem 22].

**Lemma 4.** *1. For $i \in \{0,1,2\}$ let $\overline{\mathrm{CT}}_\mathrm{i}$ be the condition obtained from $\mathrm{CT}_\mathrm{i}$ and let $\overline{\mathrm{SCT}}_0$ be the condition obtained from $SCT_0$ by replacing $\beta$ and $\nu$ by $\tau$ and $\theta$, respectively. Then $\overline{\mathrm{CT}}_\mathrm{i} \iff \mathrm{CT}_\mathrm{i}$ and $\overline{\mathrm{SCT}}_0 \iff \mathrm{SCT}_0$.*

*2. Let $\widetilde{\mathbf{X}} = (X, \tau, \widetilde{\beta}, \widetilde{\nu})$ be a computable topological space equivalent to $\mathbf{X} = (X, \tau, \beta, \nu)$ [Weihrauch and Grubba 2009, Definition 21]. Then each separation axiom from Definitions 2 and 3 for $\mathbf{X}$ is equivalent to the corresponding axiom for $\widetilde{\mathbf{X}}$.*

**Proof:** 1. $\mathrm{CT}_\mathrm{i}$ implies $\overline{\mathrm{CT}}_\mathrm{i}$ since $\nu \leq \theta$. For the other direction observe that the multi-function $h : X \times \tau \rightrightarrows \beta$ such that $U \in h(x, W) \iff x \in U \subseteq W$ is $(\delta, \theta, \nu)$-computable. The argument is valid also for $\mathrm{SCT}_0$.

2. Since $\mathbf{X}$ and $\widetilde{\mathbf{X}}$ are equivalent, there are computable functions $g, \widetilde{g} :\subseteq \Sigma^* \to \Sigma^\omega$ such that $\nu(u) = \widetilde{\theta} \circ g(u)$ and $\widetilde{\nu}(u) = \theta \circ \widetilde{g}(u)$. Furthermore, $\delta \equiv \widetilde{\delta}$ and $\theta \equiv \widetilde{\theta}$ by [Weihrauch and Grubba 2009, Theorem 22]. Since equivalent representations induce the same computability, $\overline{\mathrm{CT}}_\mathrm{i} \iff \widetilde{\overline{\mathrm{CT}}}_\mathrm{i}$ and $\overline{\mathrm{SCT}}_0 \iff \widetilde{\overline{\mathrm{SCT}}}_0$, hence by 1, $\mathrm{CT}_\mathrm{i} \iff \widetilde{\mathrm{CT}}_\mathrm{i}$ (for $i \in \{0,1,2\}$) and $\mathrm{SCT}_0 \iff \widetilde{\mathrm{SCT}}_0$.

$\mathbf{CT'_i}$ for $i = 0,1,2$: Below we will prove $\mathrm{CT}_\mathrm{i} \iff \mathrm{CT}'_\mathrm{i}$. Apply 1. of this lemma.

$\mathbf{WCT_0}$: Assume $\mathrm{SCT}_0$. Let $\widetilde{H} := \{(\widetilde{u}, \widetilde{v}) \mid (\exists (u,v) \in H)(\widetilde{u} \ll g(u) \wedge \widetilde{v} \ll g(v))\}$. Then $\widetilde{\mathrm{SCT}}_0$ can be shown straightforwardly. By symmetry, $\widetilde{\mathrm{SCT}}_2 \implies \mathrm{SCT}_2$.

$\mathbf{SCT_2}$: Use the same argument as for $\mathrm{WCT}_0$. □

## 4 Implications

In this section we prove the implications between the separation properties, in the next section we prove by counterexamples that almost all the implications are proper.

**Theorem 5.**

1. $\mathrm{SCT}_2 \Longrightarrow \mathrm{CT}_2 \Longrightarrow \mathrm{CT}_0 \Longrightarrow \mathrm{WCT}_0$,
2. $\mathrm{CT}_2 \iff \mathrm{CT}_2' \iff \mathrm{CT}_1 \iff \mathrm{CT}_1'$,
3. $\mathrm{CT}_0 \iff \mathrm{SCT}_0 \iff \mathrm{CT}_0'$,

**Proof:** $\mathbf{SCT_2 \Longrightarrow CT_2 \Longrightarrow CT_1 \Longrightarrow SCT_0 \Longrightarrow CT_0}$: Straightforward.

$\mathbf{CT_0' \Longrightarrow WCT_0}$: Straightforward.

$\mathbf{CT_0 \Longrightarrow SCT_0}$: The information wether $x \in U$ or wether $y \in U$ can be obtained from $x$, $y$ and $U \in t_0(x, y)$. If $x \neq y$ then either $x \in U$ or $y \in U$. Since the relation "$z \in U$" is $(\delta, \nu)$-r.e. we can answer the question by trying to prove $x \in U$ and $y \in U$ simultaneously.

$\mathbf{CT_0' \Longrightarrow SCT_0}$: There is a machine that on input $(p, q) \in \Sigma^\omega \times \Sigma^\omega$ first searches for $(w, u, v) \in H$ such that $u \ll p$ and $v \ll q$ and then prints $\langle w, u\rangle$ if $\nu_{\mathbb{N}}(w) = 1$ and $\langle w, v\rangle$, otherwise. Then $f_M$ realizes the function $t_0$.

$\mathbf{SCT_0 \Longrightarrow CT_0'}$: By [Weihrauch and Grubba 2009, Theorem 11] there is a computable function $g :\subseteq \Sigma^* \to \Sigma^\omega$ such that $\bigcap \nu^{\mathrm{fs}}(w) = \theta \circ g(w)$.

Let $M$ be a machine realizing the multi-function $t_0^s$. There is a machine $N$ that halts on input $(w, u, v) \in (\Sigma^*)^3$ if and only if there are words $r, s \in \mathrm{dom}(\nu^{\mathrm{fs}})$, some $u_1 \in \mathrm{dom}(\nu)$ and some $t \le \min(|r|, |s|)$ such that $M$ on input $(r1^\omega, s1^\omega)$ halts in $t$ steps with result $\langle w, u_1\rangle$ and

$$\left\{\begin{array}{ll} u \ll g(\,\widehat{r}\,\iota(u_1)) \wedge v \ll g(s) & \text{if } \nu_{\mathbb{N}}(w) = 1, \\ u \ll g(r) \ \wedge\ v \ll g(\widehat{s}\,\iota(u_1)) & \text{if } \nu_{\mathbb{N}}(w) = 2. \end{array}\right\}$$

Let $H := \mathrm{dom}(f_N)$.

For showing (4) assume $\delta(p) = x \neq y = \delta(q)$. Then there are $t, w, u_1$ such that the machine $M$ halts on input $(p, q)$ in $t$ steps with result $\langle w, u_1\rangle$ such that either $(\nu_{\mathbb{N}}(w) = 1,\ x \in \nu(u_1),\ y \notin \nu(u_1))$ or $(\nu_{\mathbb{N}}(w) = 2,\ x \notin \nu(u_1),\ y \in \nu(u_1))$. Suppose, $\nu_{\mathbb{N}}(w) = 1$. Let $r := p^{<t}$ and $s := q^{<t}$. Then $M$ also on input $(r1^\omega, s1^\omega)$ halts in $t$ steps with result $\langle w, u_1\rangle$. Since $\delta(p) = x$, $x \in \nu(u_1)$ and $\delta(q) = y$, $x \in \bigcap \nu^{\mathrm{fs}}(\widehat{r}\,\iota(u_1))$ and $y \in \bigcap \nu^{\mathrm{fs}}(s)$ [Weihrauch and Grubba 2009, Section 2 and Lemma 10], hence there are $u, v$ such that $u \ll g(\widehat{r}\,\iota(u_1))$, $x \in \nu(u)$, $v \ll g(s)$ and $y \in \nu(v)$. Therefore, there is some $(w, u, v) \in H$ such that $x \in \nu(u)$ and $y \in \nu(v)$. The argument is similar for $\nu_{\mathbb{N}}(w) = 2$. Thus (4) is proved.

For showing (5) let

$$(w, u, v) \in H,\ \nu_{\mathbb{N}}(w) = 1,\ x \in \nu(u),\ y \in \nu(u) \cap \nu(v) \text{ and } x \neq y.$$

By the definition of $H$ there are $r, s, t$ and $u_1$ such that $t \leq \min(|r|, |s|)$ and $M$ on input $(r1^\omega, s1^\omega)$ halts in $t$ steps with result $\langle w, u_1\rangle$ and $u \ll g(\widehat{r}\,\iota(u_1))$ and $v \ll g(s)$. Therefore, $x \in \nu(u) \subseteq \delta[r\Sigma^\omega] \cap \nu(u_1)$ and $y \in \nu(v) \subseteq \delta[s\Sigma^\omega]$. By ($\mathrm{SCT}_0$), $x \in \nu(u_1)$ and $y \notin \nu(u_1)$. On the other hand, $y \in \nu(u) \subseteq \nu(u_1)$ (contradiction). Therefore, if $(w, u, v) \in H$, $\nu_\mathbb{N}(w) = 1$, $x \in \nu(u)$ and $y \in \nu(u) \cap \nu(v)$, then $x = y$, hence,

$$((w, u, v) \in H,\ \nu_\mathbb{N}(w) = 1 \text{ and } \nu(u) \cap \nu(v) \neq \emptyset) \implies (\exists x)\, \nu(u) = \{x\} \subseteq \nu(v).$$

This shows (5) for the case $\nu_\mathbb{N}(w) = 1$. For the case $\nu_\mathbb{N}(w) = 2$ the argument is similar.

$\mathbf{CT_1} \iff \mathbf{CT'_1}$: This is the special case of $\mathbf{SCT_0} \iff \mathbf{CT'_0}$ where $\nu_\mathbb{N}(w) = 1$ in all cases.

$\mathbf{CT'_2} \Longrightarrow \mathbf{CT_2}$: Let $M$ be a machine which on input $(p, q)$ searches for some $(u, v) \in H$ such that $u \ll p$ and $v \ll q$ and writes $\langle u, v\rangle$ if the search is successful and diverges otherwise. Suppose $\delta(p) = x \neq y = \delta(q)$. By (8) on input $(p, q)$ the machine $M$ finds some $(u, v) \in H$ such that $x \in \nu(u) \wedge y \in \nu(v)$. By (9), $\nu(u) \cap \nu(v) = \emptyset$. Therefore, $f_M$ realizes $t_2$.

$\mathbf{CT'_1} \Longrightarrow \mathbf{CT'_2}$: Assume $\mathrm{CT}'_1$. Since $\mathbf{X}$ is a computable topological space, there is a computable function $g$ such that $\nu(u) \cap \nu(v) = \theta \circ g(u, v)$. Let $H$ be the r.e. set from $\mathrm{CT}'_1$ . Let

$$H_2 := \{(\overline{u}, \overline{v}) \mid \overline{u} \ll g(u, v'), \overline{v} \ll g(u', v) \ \text{ for some } \ (u, v), (u', v') \in H\}.$$

$H_2$ is r.e since $H$ is r.e. We prove (8) and (9) for $H_2$.

Suppose $x \neq y$. By (6) there are $(u, v)$, $(u', v') \in H$ such that $x \in \nu(u)$, $y \in \nu(v)$, $y \in \nu(u')$ and $x \in \nu(v')$. Since $x \in \nu(u) \cap \nu(v')$ and $y \in \nu(u') \cap \nu(v)$, there is some $(\overline{u}, \overline{v}) \in H_2$ such that $x \in \nu(\overline{u})$ and $y \in \nu(\overline{v})$. Therefore, (8) holds for $H_2$.

Suppose $(\overline{u}, \overline{v}) \in H_2$ and $\nu(\overline{u}) \cap \nu(\overline{v}) \neq \emptyset$. Then there are $(u, v)$, $(u', v') \in H$ such that $\nu(\overline{u}) \subseteq \nu(u) \cap \nu(v')$ and $\nu(\overline{v}) \subseteq \nu(u') \cap \nu(v)$. Since $\nu(\overline{u}) \cap \nu(\overline{v}) \neq \emptyset$, $\nu(u) \cap \nu(v) \neq \emptyset$ and $\nu(u') \cap \nu(v') \neq \emptyset$. By (7) for some $x, y \in X$, $\nu(u) = \{x\} \subseteq \nu(v)$ and $\nu(u') = \{y\} \subseteq \nu(v')$ , hence $\nu(\overline{u}) \subseteq \{x\}$ and $\nu(\overline{v}) \subseteq \{y\}$. Since $\nu(\overline{u}) \cap \nu(\overline{v}) \neq \emptyset$, $\nu(\overline{u}) = \{x\} = \nu(\overline{v})$. Therefore, (9) holds for $H_2$.

$\mathbf{CT'_0} \Longrightarrow \mathbf{WCT_0}$: Obvious.

The remaining implications follow by transitivity. □

Surprisingly, computable $T_1$ is the same as computable $T_2$. The hierarchy between $\mathrm{WCT}_0$ and $\mathrm{SCT}_2$ collapses for spaces without isolated points.

**Corollary 6.** *If $\{x\}$ is not open for all $x \in X$ then* $\mathrm{WCT}_0 \Longrightarrow \mathrm{SCT}_2$

**Proof:** If $\{x\}$ is not open for all $x \in X$ then $\mathrm{WCT}_0$ is equivalent to $\mathrm{SCT}_2$. □

The space $\mathbf{R}_<$ from Example 1.2 is not $T_2$ since every pair of non-empty open sets has non-empty intersection. By Corollary 6 the space $\mathbf{R}_<$ is not even $WCT_0$. The outer representation $\delta^- :\subseteq \Sigma^\omega \to X$ of points is defined by $\delta^-(p) = x \iff \theta(p) = X \setminus \overline{\{x\}}$ [Weihrauch and Grubba 2009, Definition 5]. The 2nd statement below has been proved in [Xu and Grubba 2009] for spaces with non-empty base elements.

**Theorem 7.** *For computable topological spaces* $\mathbf{X}$*,*

1. $\mathbf{X}$ *is* $SCT_2$*, if* $\mathbf{X}$ *is* $T_2$ *and* $\{(u,v) \mid \nu(u) \cap \nu(v) = \emptyset\}$ *is r.e.,*
2. $\mathbf{X}$ *is* $SCT_2$ *iff* $x \neq y$ *is* $(\delta,\delta)$*-r.e..*
3. $\mathbf{X}$ *is* $SCT_2$ *iff* $\delta \leq \delta^-$*.*

**Proof:**

1. Let $H := \{(u,v) \mid \nu(u) \cap \nu(v) = \emptyset\}$.

2. $\Longrightarrow$: By (10,11), for all $x, y \in X$, $x \neq y \iff (\exists (u,v) \in H)\,(x \in \nu(u) \;\wedge\; y \in \nu(v))$. Since "$x \in \nu(u)$" is $(\delta,\nu)$-r.e. [Weihrauch and Grubba 2009], $x \neq y$ is $(\delta,\delta)$-r.e.

$\Longleftarrow$: By [Weihrauch and Grubba 2009, Theorem 11] there is a computable function $g$ such that $\bigcap \nu^{\rm fs}(w) = \theta \circ g(w)$ for all $w \in {\rm dom}(\nu^{\rm fs})$. Suppose that $x \neq y$ is $(\delta,\delta)$-r.e. Then there is a machine $M$ that halts on input $(p,q) \in {\rm dom}(\delta) \times {\rm dom}(\delta)$ iff $\delta(p) \neq \delta(q)$. There is a machine $N$ that halts on input $(u,v)$ iff $u, v \in {\rm dom}(\nu)$ and there are $u_0, v_0 \in {\rm dom}(\nu^{\rm fs})$ such that $M$ halts on input $(u_0 1^\omega, v_0 1^\omega)$ in at most $\min(|u_0|, |v_0|)$ steps and $u \ll g(u_0)$ and $v \ll g(v_0)$. Let $H := {\rm dom}(f_N)$. We must show (10) and (11).

Suppose that $x \neq y$. There are $p, q$ such that $x = \delta(p)$ and $y = \delta(q)$. Then $M$ halts on input $(p,q)$, hence there are $u_0 \ll p$ and $v_0 \ll q$ such that $M$ halts on input $(u_0 1^\omega, v_0 1^\omega)$ in at most $\min(|u_0|, |v_0|)$ steps. There are $u \ll g(u_0)$ and $v \ll g(v_0)$ such that $x \in \nu(u)$ and $y \in \nu(v)$. By definition, $N$ halts on input $(u,v)$. This proves (10).

Suppose $(u,v) \in H$. Then there are $u_0, v_0 \in {\rm dom}(\nu^{\rm fs})$ such that $M$ halts on input $(u_0 1^\omega, v_0 1^\omega)$ in at most $\min(|u_0|, |v_0|)$ steps and $u \ll g(u_0)$ and $v \ll g(v_0)$.

If $\nu(u) = \emptyset$ or $\nu(v) = \emptyset$ then $\nu(u) \cap \nu(v) = \emptyset$. Assume $x \in \nu(u)$ and $y \in \nu(v)$. Then $x = \delta(u_o p)$ and $y = \delta(v_0 q)$ for some $p, q \in \Sigma^\omega$. Since $M$ must halt also on input $(u_0 p, v_0 q)$, $x \neq y$. Therefore $\nu(u) \cap \nu(v) = \emptyset$. This proves (11).

3. For every open set $W$,

$$W \cap B = \emptyset \iff W \cap \overline{B} = \emptyset. \tag{12}$$

Suppose SCT$_2$. Since $\delta(p) \in \nu(u)$ is r.e. [Weihrauch and Grubba 2009, Theorem 13.1] and $H$ is r.e. by assumption (Definition 3), there is a computable function $h :\subseteq \Sigma^\omega \to \Sigma^\omega$ such that for all $p \in {\rm dom}(\delta)$, $v \ll h(p)$ iff $(\exists u)\,(\delta(p) \in \nu(u)$

and $(u,v) \in H$). Suppose $\delta(p) = x$. For every $y \neq x$ there is some $(u,v) \in H$ such that $x \in \nu(u)$ and $y \in \nu(v)$. Therefore, $\theta \circ h(p) = X \setminus \{x\}$.

By (12), $\theta \circ h(p) \subseteq X \setminus \overline{\{x\}} \subseteq X \setminus \{x\} = \theta \circ h(p)$, hence $\delta^- \circ h(p) = x$. Therefore, the function $h$ translates $\delta$ to $\delta^-$.

On the other hand let $h$ be a computable function translating from $\delta$ to $\delta^-$. Let $\delta(p) = x$ and $\delta(q) = y$. Since $\mathbf{X}$ is $T_0$,

$x \neq y \iff (\exists W \in \tau)(x \in W \;\wedge\; y \notin W)$ or $(\exists W \in \tau)(x \notin W \;\wedge\; y \in W)$.

Since $y \notin W \iff W \subseteq X \setminus \overline{\{y\}}$ by (12), and $\delta^- \circ h(q) = y$,

$$\begin{aligned}(\exists W \in \tau)(x \in W \;\wedge\; y \notin W) &\iff (\exists W \in \tau)(x \in W \;\wedge\; W \subseteq X \setminus \overline{\{y\}})\\ &\iff x \in X \setminus \overline{\{y\}})\\ &\iff \delta(p) \in \theta \circ h(q)\end{aligned},$$

and correspondingly $(\exists W \in \tau)(x \notin W \;\wedge\; y \in W) \iff \delta(q) \in \theta \circ h(p)$. Since $x \in V$ is $(\delta,\theta)$-r.e. [Weihrauch and Grubba 2009, Corollary 14], there is a machine that halts on input $(p,q)$ iff $\delta(p) \neq \delta(q)$. By (2) of this theorem, the space is $SCT_2$. □

## 5 Counterexamples

We show by counterexamples that a number of implications for the computable separation axioms for computable separable spaces are not true in general. A topological space is discrete iff every singleton $\{x\}$ is open iff every subset $B \subseteq X$ is open. Every discrete space is $T_i$ for $i = 0, 1, 2$. Let "D" be the axiom stating that the topological space is discrete. In the following examples let $(a_i)_{i\in\mathbb{N}}$, $(b_i)_{i\in\mathbb{N}}$, ..., $(e_i)_{i\in\mathbb{N}}$ be injective families with pairwise disjoint ranges and let $\{0,1,\ldots,7\} \subseteq \Sigma$.

*Example 2.* ($D$ but not $WCT_0$) Let $X := \{a_i, b_i, c_i, d_i, e_i \mid i \in \mathbb{N}\}$ and let $\tau$ be the discrete topology on $X$. For all $i \in \mathbb{N}$ let $\nu(0^i5) = \{c_i\}$, $\nu(0^i6) = \{d_i\}$, $\nu(0^i7) = \{e_i\}$. Let $A \subseteq \mathbb{N}$ be some non-r.e. set and define $\nu(0^i1), \ldots, \nu(0^i4)$ by the following table.

| | $\nu(0^i1)$ | $\nu(0^i2)$ | $\nu(0^i3)$ | $\nu(0^i4)$ |
|---|---|---|---|---|
| $i \in A$ | $\{a_i\}$ | $\{b_i\}$ | $\{c_i, d_i\}$ | $\{d_i, e_i\}$ |
| $i \notin A$ | $\{c_i, d_i\}$ | $\{d_i, e_i\}$ | $\{a_i\}$ | $\{b_i\}$ |

The function $\nu$ defined so far is a notation of a base of the discrete topology on $X$. In order to make intersection computable we extend $\nu$ by adding names of intersections for 2 and 3 different names defined so far. For all $i \in \mathbb{N}$ and all $k, l, m \in \{1, \ldots, 7\}$ such that $k \neq l$, $k \neq m$ and $l \neq m$ let $\nu(0^ikl) := \nu(0^ik) \cap \nu(0^il)$ and $\nu(0^iklm) := \nu(0^ik) \cap \nu(0^il) \cap \nu(0^im)$. Let $\beta := \mathrm{range}(\nu)$. Since for each $i$ the intersection of this kind of more than 3 elements can be reduced to the intersection of 3 elements, $\mathbf{X} := (X, \tau, \beta, \nu)$ is a computable topological space.

Suppose $\mathbf{X}$ is $WCT_0$. Let $H \subseteq \mathrm{dom}(\nu) \times \mathrm{dom}(\nu)$ be an r.e. set such that (2) and (3). By (2) for $i \in A$ there must be some $(u,v) \in H$ such that $a_i \in \nu(u)$

and $b_i \in \nu(v)$. Then $u = 0^i1$ and $v = 0^i2$, hence $(0^i1, 0^i2) \in H$. For $i \notin A$, $\nu(0^i1) = \{c_i, d_i\}$ and $\nu(0^i2) = \{d_i, e_i\}$. Since (3) is violated, $(0^i1, 0^i2) \notin H$. Therefore, $i \in A$ iff $(0^i1, 0^i2) \in H$. Since $H$ is r.e., the set $A$ must be r.e. Contradiction. □

*Example 3.* ($D + WCT_0$ but not $CT_0$) Let $A \subseteq \mathbb{N}$ be some non-r.e. set. Let $X := \{a_i, b_i \mid i \in \mathbb{N}\}$ and let $\tau$ be the discrete topology on $X$. Below we will define sets $B, C, D \subseteq \mathbb{N}$ such that $\{A, B, C, D\}$ is a partition of $\mathbb{N}$. Define a notation $\nu$ of a basis $\beta$ of the topology as follows.

| | $0^i1$ | $0^i2$ | $0^i3$ | $0^i12$ | $0^i13$ | $0^i23$ |
|---|---|---|---|---|---|---|
| $i \in A \cup D$ | $\{a_i\}$ | $\{b_i\}$ | $\emptyset$ | $\emptyset$ | $\emptyset$ | $\emptyset$ |
| $i \in B$ | $\{a_i\}$ | $\{a_i, b_i\}$ | $\{b_i\}$ | $\{a_i\}$ | $\emptyset$ | $\{b_i\}$ |
| $i \in C$ | $\{a_i, b_i\}$ | $\{b_i\}$ | $\{a_i\}$ | $\{b_i\}$ | $\{a_i\}$ | $\emptyset$ |

Then $\nu(u) \cap \nu(v) = \nu \circ g(u, v)$ for some computable function $g$, since $\nu(0^ik) \cap \nu(0^im) = \nu(0^ikm)$ for $k \neq m$. Therefore $\mathbf{X} := (X, \tau, \beta, \nu)$ is a computable topological space. Let $H := \{(0^ik, 0^jl) \mid i, j \in \mathbb{N};\ k, l \in \{1, 2\};\ (i \neq j \vee k \neq l)\}$. Then H satisfies (2) and (3) for the space $\mathbf{X}$. Therefore, $\mathbf{X}$ is a WCT$_0$-space.

We will define $B$ and $C$ such that $\mathbf{X}$ is not $SCT_0$. Let $l, r \in \Sigma^*$ such that $\nu_{\mathbb{N}}(l) = 1$ and $\nu_{\mathbb{N}}(r) = 2$. We assume w.l.o.g. that $\nu_{\mathbb{N}}$ is injective. For $i \in \mathbb{N}$ let

$$S_i := \{\langle l, 0^i1\rangle,\ \langle r, 0^i3\rangle,\ \langle l, 0^i12\rangle,\ \langle r, 0^i23\rangle\},$$
$$T_i := \{\langle r, 0^i2\rangle,\ \langle l, 0^i3\rangle,\ \langle r, 0^i12\rangle,\ \langle l, 0^i13\rangle\}.$$

Suppose, the function $f :\subseteq \Sigma^\omega \times \Sigma^\omega \to \Sigma^*$ realizes the separation function $t_0^s$ for $\mathbf{X}$. If $\delta(p) = a_i$ and $\delta(q) = b_i$ then

$$f(p, q) \in \begin{cases} S_i \text{ if } & i \in B \\ T_i \text{ if } & i \in C \end{cases} \tag{13}$$

since $\nu(u)$ must be either $\{a_i\}$ or $\{b_i\}$ if $f(p, q) = \langle w, u\rangle$. Notice that $S_i \cap T_i = \emptyset$.

For all $i \in \mathbb{N}$ define $p_i, q_i \in \Sigma^\omega$ by $p_i := \iota(0^i1)\iota(0^i1)\iota(0^i1)\ldots$ and $q_i := \iota(0^i2)\iota(0^i2)\iota(0^i2)\ldots$. Let $F$ be the set of all computable functions $f :\subseteq \Sigma^\omega \times \Sigma^\omega \to \Sigma^*$ such that $f(p_i, q_i)$ exists for all $i \in A$. Consider $f \in F$. Then $f' : i \mapsto f(p_i, q_i)$ is computable such that $A \subseteq \mathrm{dom}(f')$. Since $A$ is not r.e. and $\mathrm{dom}(f')$ is r.e., $\mathrm{dom}(f') \setminus A$ is infinite. Since $F$ is countable, there is a bijective function $g : E \to F$ for some $E \subseteq \mathbb{N}$ such that $i \in \mathrm{dom}(g_i') \setminus A$ for all $i \in E$ ($g_i := g(i)$). Then $A \cap E = \emptyset$. Notice that $g_i(p_i, q_i)$ exists for all $i \in E$.

For each $i \in E$ we put $i$ to $B$ or $C$ in such a way that $g_i$ does not realize the separating function $t_0^s$ for SCT$_0$. Let

$$\begin{aligned} B &:= \{i \in E \mid g_i(p_i, q_i) \notin S_i\}, \\ C &:= \{i \in E \mid g_i(p_i, q_i) \in S_i\}, \end{aligned} \tag{14}$$

and $D := \mathbb{N} \setminus (A \cup B \cup C)$. Since $A \cap E = \emptyset$, $E = B \cup C$ and $B \cap C = \emptyset$, $\{A, B, C, D\}$ is a partition of $\mathbb{N}$.

Suppose some computable function $f$ realizes $t_0^s$. Since $\delta(p_i) = a_i$ and $\delta(q_i) = b_i$ for $i \in A$, $f(p_i, q_i)$ exists for all $i \in A$, hence $f = g_i$ for some $i \in E$.

Since $g_i$ realizes $t_0^s$, $g_i(p_i, q_i) \in S_i \iff i \in B$ by(13). On the other hand, $g_i(p_i, q_i) \in S_i \iff i \notin B$ by the definition of $B$ (14). From this contradiction we conclude that $\mathbf{X}$ is not $SCT_0$. By Theorem 5, $\mathbf{X}$ is not $CT_0$. □

*Example 4.* ($D$ and $CT_0$ but not $CT_1$) Let $A \subseteq \mathbb{N}$ be some non-r.e. set. Let $X := \{a_i, b_i \mid i \in \mathbb{N}\}$ and let $\tau$ be the discrete topology on $X$. Below we will define sets $B, C, D \subseteq \mathbb{N}$ such that $\{A, B, C, D\}$ is a partition of $\mathbb{N}$. For $i \in \mathbb{N}$ define $\nu(0^i1), \ldots, \nu(0^i4)$ as follows.

| | $0^i1$ | $0^i2$ | $0^i3$ | $0^i4$ |
|---|---|---|---|---|
| $i \in A \cup D$ | $\{a_i\}$ | $\{b_i\}$ | $\emptyset$ | $\emptyset$ |
| $i \in B$ | $\{a_i\}$ | $\{a_i, b_i\}$ | $\emptyset$ | $\{b_i\}$ |
| $i \in C$ | $\{a_i\}$ | $\{a_i, b_i\}$ | $\{b_i\}$ | $\emptyset$ |

For $k, m \in \{1, 2, 3, 4\}$, $k \neq m$, define $\nu(0^ikm) := \nu(0^ik) \cap \nu(0^im)$. Let $\beta := \text{range}(\nu)$. Since for each $i$ and pairwise different $k, l, m$, $\nu(0^ik) \cap \nu(0^im) \cap \nu(0^im) = \emptyset$, $\mathbf{X} := (X, \tau, \beta, \nu)$ is a computable topological space. Let $P_i := \{0^ik, 0^ikl \mid k, l \in \{1, 2, 3, 4\}\}$. If $\delta(p) = x$, then $x \in \{a_i, b_i\}$ for some $i \in \mathbb{N}$, hence $u \in P_i$ for all $u \ll p$, since by definition $u \ll p \iff x \in \nu(u)$.

We show that the space $\mathbf{X}$ is $CT_0$. There is a machine that on input $p, q \in \Sigma^\omega$ searches for words $u \ll p$ and $v \ll q$. If $u, v \in P_i$ for some $i$ then the machine writes $0^i1$, if $u \in P_i$ and $v \in P_j$ for some $i \neq j$, then it writes $u$. Suppose $\delta(p) \neq \delta(q)$.

Case: $u, v \in P_i$ for some $i$: Then $\{\delta(p), \delta(q)\} = \{a_i, b_i\}$, hence $\delta(p) \in \nu(0^i1) = \{a_i\}$ or $\delta(q) \in \nu(0^i1) = \{a_i\}$. Therefore, $(\delta(p) \in \nu(u)$ and $\delta(q) \notin \nu(u))$ or $(\delta(p) \notin \nu(u)$ and $\delta(q) \in \nu(u))$.

Case: $u \in P_i$ and $v \in P_j$ for some $i \neq j$: Then $\delta(p) \in \nu(u) \subseteq \{a_i, b_i\}$ and $\delta(q) \in \nu(v) \subseteq \{a_j, b_j\}$. Therefore, $\delta(p) \in \nu(u)$ and $\delta(q) \notin \nu(u)$.

In summary, $\mathbf{X}$ is $CT_0$.

We show that $\mathbf{X}$ is not $CT_2$. For all $i \in \mathbb{N}$ define $p_i, q_i \in \Sigma^\omega$ by

$$p_i := \iota(0^i1)\iota(0^i1)\iota(0^i1)\ldots$$
$$q_i := \iota(0^i2)\iota(0^i2)\iota(0^i2)\ldots.$$

Let $F$ be the set of all computable functions $f :\subseteq \Sigma^\omega \times \Sigma^\omega \to \Sigma^*$ such that $f(p_i, q_i)$ exists for all $i \in A$. Consider $f \in F$. Then $f' : i \mapsto f(p_i, q_i)$ is computable such that $A \subseteq \text{dom}(f')$. Since $A$ is not r.e. and $\text{dom}(f')$ is r.e., $\text{dom}(f') \setminus A$ is infinite. Since $F$ is countable, there is a bijective function $g : E \to F$ for some $E\subseteq\mathbb{N}$ such that $i \in \mathrm{dom}(g'_i) \setminus A$ for all $i \in E$ ($g_i := g(i)$). Then $A \cap E = \emptyset$ and $g_i(p_i, q_i) \in \Sigma^*$ exists for all $i \in E$.

For each $i \in E$ we put $i$ to $B$ or $C$ in such a way that $g_i$ does not realize the separating function $t_2$ for $\mathrm{CT}_2$. For $i \in E$ let

$$i \in B :\Longleftrightarrow \neg(\exists u \in \Sigma^*,\ v \in \{0^i4, 0^i24\})g_i(p_i, q_i) = \langle u, v\rangle, \tag{15}$$

$C := E \setminus B$ and $D := \mathbb{N} \setminus (A \cup B \cup C)$. Then $\{A, B, C, D\}$ is a partition of $\mathbb{N}$.

Suppose there is some computable function $f$ that realizes the separating function $t_2$ for $\mathrm{CT}_2$. Since $\delta(p_i) = a_i$ and $\delta(q_i) = b_i$ for $i \in A$, $f(p_i, q_i)$ exists for all $i \in A$. Therefore, $f = g_i$ for some $i \in E$, hence $g_i(p_i, q_i)$ exists.

There are $w_1, w_2 \in \Sigma^*$ such that $g_i(p_i, q_i) = g_i(w_1p, w_2q)$ for all $p, q \in \Sigma^\omega$.

Suppose $i \in B$. There are $p, q$ such that $\delta(w_1p) = a_i$ and $\delta(w_2q) = b_i$. Since $f = g_i$ realizes $t_2$, there are $u, v \in \Sigma^*$ such that $g_i(p_i, q_i) = g_i(w_1p, w_2q) = \langle u, v\rangle$ and $\nu(v) = \{b_i\}$, hence $v \in \{0^i4, 0^i24\}$. But then $i \notin B$ by (15). Contradiction.

Suppose $i \in C = E \setminus B$. Again there are $p, q$ such that $\delta(w_1p) = a_i$ and $\delta(w_2q) = b_i$. Also, since $f = g_i$ realizes $t_2$, there are $u, v \in \Sigma^*$ such that $g_i(p_i, q_i) = g_i(w_1p, w_2q) = \langle u, v\rangle$ and $\nu(v) = \{b_i\}$, hence $v \in \{0^i3, 0^i23\}$. But then $i \in B$ by (15). Contradiction.

From this contradiction we conclude that $\mathbf{X}$ is not $CT_2$. By Theorem 5, $\mathbf{X}$ is not $CT_1$. □

*Example 5.* ($D$ and $CT_2$ but not $SCT_2$) Let $A\subseteq\mathbb{N}$ be an r.e. set with non-r.e. complement. Define a notation $\nu$ by

$$\nu(0^i1) := \{a_i\}, \nu(0^i2) := \{a_i\} \text{ for } i \in A,$$
$$\nu(0^i1) := \{a_i\}, \nu(0^i2) := \{b_i\} \text{ for } i \notin A$$

for all $i \in \mathbb{N}$. Then $\nu$ is a notation of a base $\beta$ of a topology (the discrete topology) $\tau$ on a subset $X\subseteq\mathbb{N}$ such that $\mathbf{X} = (X, \tau, \beta, \nu)$ is a computable topological space.

The space $\mathbf{X}$ is $T_2$ since it is discrete. It is $CT_2$ but not $SCT_2$: The set $H := \{(0^ik, 0^jl) \mid i, j \in \mathbb{N},\ k, l \in \{1, 2\}\}$ satisfies $\mathrm{CT}'_2$. By Theorem 5 the space is $CT_2$. Suppose $\mathrm{SCT}_2$. Let $H$ be the r.e. set for $\mathrm{SCT}_2$. By (10), $i \notin A \Longrightarrow (0^i1, 0^i2) \in H$ and by (11), $i \in A \Longrightarrow (0^i1, 0^i2) \notin H$. Since $H$ is r.e., the complement of $A$ must be r.e. (contradiction). Notice that $x \neq y$ is not $(\delta, \delta)$-r.e., see Theorem 7.2. □

We summarize the counterexamples as follows.

**Theorem 8.** *For computable topological spaces,*

$$\mathrm{T}_0 \not\Longrightarrow \mathrm{WCT}_0 \qquad \textit{(Example 1.2)} \qquad (16)$$

$$\mathrm{T}_1 \not\Longrightarrow \mathrm{WCT}_0 \qquad \textit{(Example 1.4)} \qquad (17)$$

$$\mathrm{D} \not\Longrightarrow \mathrm{WCT}_0 \qquad \textit{(Example 2;)} \qquad (18)$$

$$\mathrm{D} + \mathrm{WCT}_0 \not\Longrightarrow \mathrm{CT}_0 \qquad \textit{(Example 3)} \qquad (19)$$

$$\mathrm{D} + \mathrm{CT}_0 \not\Longrightarrow \mathrm{CT}_1 \qquad \textit{(Example 4)} \qquad (20)$$

$$\mathrm{D} + \mathrm{CT}_2 \not\Longrightarrow \mathrm{SCT}_2 \qquad \textit{(Example 5)} \qquad (21)$$

Since $\mathrm{D} \Longrightarrow \mathrm{T}_2 \Longrightarrow \mathrm{T}_1 \Longrightarrow \mathrm{T}_0$, (16), (17) as well as $\mathrm{T}_2 \not\Longrightarrow \mathrm{CT}_2$ follow from (18) by Theorem 5. Further results can be obtained in the same way.

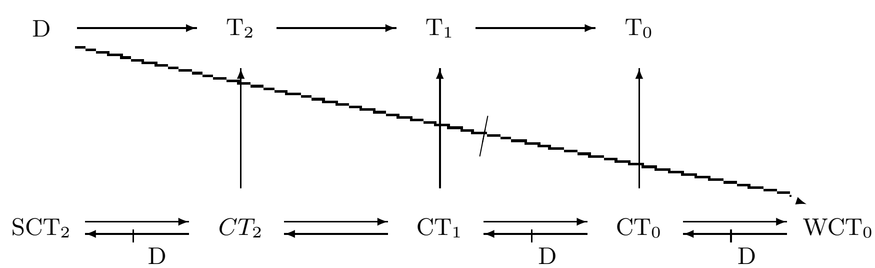


**Figure 1:** The relation between computable $T_0$-, $T_1$-, and $T_2$-separation.

Figure 1 visualizes the relations between the computable versions of $T_i$ for $i = 0, 1, 2$ from Definitions 2 and 3 we have proved. "$A \longrightarrow B$" means $A \Longrightarrow B$, "$A \not\longrightarrow B$" means that we have constructed a computable topological space for which $A \wedge \neg B$, and "$A \overset{C}{\not\longrightarrow} B$" means that we have constructed a computable topological space for which $(A \wedge C) \wedge \neg B$. Remember that $\mathrm{SCT}_0 \iff \mathrm{CT}_0 \iff \mathrm{CT}_0'$ and $\mathrm{CT}_1 \iff \mathrm{CT}_1' \iff \mathrm{CT}_2 \iff \mathrm{CT}_2'$.

## 6 Separation of Compact Sets and Intuitionistic Separation

In a Hausdorff space not only different points but also disjoint compact sets can be separated by open neighborhoods [Engelking 1989]. For each of the axioms $\mathrm{CT}_2$ and $\mathrm{SCT}_2$ we introduce generalizations for separating points and compact sets and for separating compact sets and compact sets.

**Definition 9.**

$\mathrm{CT}_2^{\mathrm{pc}}$ : The multi-function $t^{\mathrm{pc}}$ is $(\delta, \kappa, [\nu, \bigcup \nu^{\mathrm{fs}}])$-computable, where $t^{\mathrm{pc}}$ maps every $x \in X$ and every compact $K$ such that $x \notin K$ to some pair $(U, W)$ of disjoint open sets such that $x \in U$ and $K \subseteq W$.

$\mathrm{CT}_2^{\mathrm{cc}}$ : The multi-function $t^{\mathrm{cc}}$ is $(\kappa, \kappa, [\bigcup \nu^{\mathrm{fs}}, \bigcup \nu^{\mathrm{fs}}])$-computable, where $t^{\mathrm{cc}}$ maps every disjoint pair $(K, L)$ of non-empty compact sets to some pair $(V, W)$ of disjoint open sets such that $K \subseteq V$ and $L \subseteq W$.

$\mathrm{SCT}_2^{\mathrm{pc}}$ : There is an r.e. set $H \in \Sigma^* \times \Sigma^*$ such that

$$\left\{ \begin{array}{l} (\forall x \in X)(\forall \text{ compact } K) \text{ such that } x \notin K \\ (\exists (u, w) \in H)(x \in \nu(u) \wedge K \subseteq \bigcup \nu^{\mathrm{fs}}(w)) \end{array} \right\} \quad \text{and} \tag{22}$$

$$(\forall (u, w) \in H)\, \nu(u) \cap \bigcup \nu^{\mathrm{fs}}(w) = \emptyset\,. \tag{23}$$

$\mathrm{SCT}_2^{\mathrm{cc}}$ : There is an r.e. set $H \in \Sigma^* \times \Sigma^*$ such that

$$\left\{ \begin{array}{l} (\forall \text{ compact } K, L) \text{ such that } K \cap L = \emptyset \\ (\exists (u, v) \in H)(K \subseteq \bigcup \nu^{\mathrm{fs}}(u) \wedge L \subseteq \bigcup \nu^{\mathrm{fs}}(v)) \end{array} \right\} \quad \text{and} \tag{24}$$

$$(\forall (u, v) \in H) \bigcup \nu^{\mathrm{fs}}(u) \cap \bigcup \nu^{\mathrm{fs}}(v) = \emptyset\,. \tag{25}$$

For the above computable separation axioms the notation $\nu$ of the base and the notation $\bigcup \nu^{\mathrm{fs}}$ of the finite unions of base elements can be replaced by the representation $\theta$ of the open sets, and the axioms are robust, that is, they do not depend on the notation $\nu$ of the base explicitly but only on the computability concept on the points induced by it [Weihrauch and Grubba 2009, Definition 21, Theorem 22].

**Lemma 10.** *1. Let* $\overline{\mathrm{CT}}_2^{\mathrm{pc}}$ *and* $\overline{\mathrm{CT}}_2^{\mathrm{cc}}$ *be the conditions obtained from* $\mathrm{CT}_2^{\mathrm{pc}}$ *and* $\mathrm{CT}_2^{\mathrm{cc}}$*, respectively, by replacing* $\nu$ *and* $\bigcup \nu^{\mathrm{fs}}$ *by* $\theta$*. Then* $\overline{\mathrm{CT}}_2^{\mathrm{pc}} \iff \mathrm{CT}_2^{\mathrm{pc}}$ *and* $\overline{\mathrm{CT}}_2^{\mathrm{cc}} \iff \mathrm{CT}_2^{\mathrm{cc}}$

*2. Let* $\widetilde{\mathbf{X}} = (X, \tau, \widetilde{\beta}, \widetilde{\nu})$ *be a computable topological space equivalent to* $\mathbf{X} = (X, \tau, \beta, \nu)$ *[Weihrauch and Grubba 2009, Definition 21]. Then the separation axioms* $\mathrm{SCT}_2^{\mathrm{pc}}$ *and* $\mathrm{SCT}_2^{\mathrm{cc}}$ *for* $\mathbf{X}$ *is equivalent to the corresponding axiom for* $\widetilde{\mathbf{X}}$*.*

**Proof:** Straightforward, see the proof of Lemma 4. □

The computable $T_2$ axioms are related as follows:

**Theorem 11.**

*1.* $\mathrm{SCT}_2^{\mathrm{cc}} \iff \mathrm{SCT}_2^{\mathrm{pc}} \iff \mathrm{SCT}_2 \Longrightarrow \mathrm{CT}_2^{\mathrm{cc}} \Longrightarrow \mathrm{CT}_2^{\mathrm{pc}} \Longrightarrow \mathrm{CT}_2$

*2. The space* $\mathbf{X}$ *from Example 5 is* $CT_2$ *but not* $CT_2^{\mathrm{pc}}$*.*

**Proof:** 1. $\mathbf{SCT_2} \Longrightarrow \mathbf{SCT_2^{pc}}$ : Since intersection on open sets is computable [Weihrauch and Grubba 2009, Theorem 11], there is a computable function $g$ such that $\theta \circ g(v) = \bigcap \nu^{\mathrm{fs}}(v)$. Let $H \subseteq \Sigma^* \times \Sigma^*$ be the r.e. set satisfying (10) and (11) from Definition 3. Let $H'$ be the set of all $(u, w) \in \Sigma^* \times \Sigma^*$ such that for some finite set $N \subseteq H$ and some word $v$, $\nu^{\rm fs}(v) = {\rm pr}_1 N$, $\nu^{\rm fs}(w) = {\rm pr}_2 N$ and $u \ll g(v)$. The set $H'$ is r.e. Suppose, $K$ is compact and $x \notin K$. By (10) for each $y \in K$ there is some $(u', v') \in H$ such that $x \in \nu(u')$, $y \in \nu(v')$ and $\nu(u') \cap \nu(v') = \emptyset$. Since $K$ is compact, there are a finite subset $N \subseteq H$ of such pairs and words $u, w$ such that $\nu^{\rm fs}(v) = {\rm pr}_1 N$, $\nu^{\rm fs}(w) = {\rm pr}_2 N$, $x \in \bigcap \nu^{\rm fs}(v)$ and $K \subseteq \bigcup \nu^{\rm fs}(w)$. Finally, there is some $u$ such that $x \in \nu(u)$ and $u \ll g(v)$. By definition, $(u, w) \in H'$. This proves (22).

Suppose $(u, w) \in H'$. Then there are some finite set $N \subseteq H$ and some word $v$ such that $\nu^{\rm fs}(v) = {\rm pr}_1 N$, $\nu^{\rm fs}(w) = {\rm pr}_2 N$ and $u \ll g(v)$. Since $\nu(u') \cap \nu(v') = \emptyset$ for all $(u', v') \in N$, $\bigcap \nu^{\rm fs}(v) \cap \bigcup \nu^{\rm fs}(w) = \emptyset$ and hence $\nu(u) \cap \bigcup \nu^{\rm fs}(w) = \emptyset$ since $\nu(u) \subseteq \bigcap \nu^{\rm fs}(v)$. This proves (23)

$\mathbf{SCT_2^{pc}} \Longrightarrow \mathbf{SCT_2^{cc}}$ : There is a computable function $f_1$ such that $\bigcup \nu^{\rm fs}(w) = \theta \circ f_1(w)$ [Weihrauch and Grubba 2009, Lemma 10]. Then the function $f_2 : \iota(v_1) \ldots \iota(v_n)$
$\mapsto \langle 1^n, f_1(v_1), \ldots, f_1(v_n) \rangle$ is computable. By [Weihrauch and Grubba 2009, Lemma 11.1] there is a computable function $f_3$ such that $\bigcap \theta^{\rm fs}(q) = \theta \circ f_3(q)$. Therefore, for the computable function $f := f_3 \circ f_2$, $\bigcup \nu^{\rm fs}(v_1) \cap \ldots \cap \bigcup \nu^{\rm fs}(v_n) = \theta \circ f(\iota(v_1) \ldots \iota(v_n))$.

Let $H \subseteq \Sigma^* \times \Sigma^*$ be the r.e. set satisfying (22) and (23) from Definition 9. Let $H'$ be the set of all pairs $(u, v)$ of words for which there are some $n$, and pairs $(u_1, v_1), \ldots, (u_n, v_n) \in H$ such that $u = \iota(u_1) \ldots \iota(u_n)$ and $v$ is a prefix of $f(\iota(v_1) \ldots \iota(v_n))$. We show that (24) and (25) are true for $H'$.

Let $K, L$ be disjoint compact sets. Then by (22) for every $y \in K$ there is some $(u, v) \in H$ such that $y \in \nu(u)$ and $L \subseteq \bigcup \nu^{\rm fs}(v)$. Since $K$ is compact there are $(u_1, v_1), \ldots, (u_n, v_n) \in H$ such that $K \subseteq \nu(u_1) \cup \ldots \cup \nu(u_n) = \bigcup \nu^{\rm fs}(u)$ for $u = \iota(u_1) \ldots \iota(u_n)$ and $L \subseteq \bigcup \nu^{\rm fs}(v_1) \cap \ldots \cap \bigcup \nu^{\rm fs}(v_n) = \theta \circ f(\iota(v_1) \ldots \iota(v_n))$. Since $L$ is compact there is some prefix $v$ of $f(\iota(v_1) \ldots \iota(v_n))$ such that $L \subseteq \bigcup \nu^{\rm fs}(v)$ $\subseteq f(\iota(v_1) \ldots \iota(v_n))$. Therefore, $(u, v) \in H'$, $K \subseteq \bigcup \nu^{\rm fs}(u)$ and $L \in \bigcup \nu^{\rm fs}(v)$. This proves (24).

Suppose, $(u, v) \in H'$. Then there are $(u_1, v_1), \ldots, (u_n, v_n) \in H$ such that $u = \iota(u_1) \ldots \iota(u_n)$ and $v$ is a prefix of $f(\iota(v_1) \ldots \iota(v_n))$. Then $(\nu(u_1) \cup \ldots \cup \nu(u_n)) \cap (\bigcup \nu^{\rm fs}(v_1) \cap \ldots \cap \bigcup \nu^{\rm fs}(v_n) = \emptyset$. Since $\bigcup \nu^{\rm fs}(u) = \nu(u_1) \cup \ldots \cup \nu(u_n)$, $\bigcup \nu^{\rm fs}(v_1) \cap \ldots \cap \bigcup \nu^{\rm fs}(v_n) = \theta \circ f(\iota(v_1) \ldots \iota(v_n))$ and $v$ is a prefix of $f(\iota(v_1) \ldots \iota(v_n))$, $\bigcup \nu^{\rm fs}(v) \subseteq f(\iota(v_1) \ldots \iota(v_n))$, hence $\bigcup \nu^{\rm fs}(u) \cap \bigcup \nu^{\rm fs}(v) = \emptyset$. This proves (25).

$\mathbf{SCT_2^{cc}} \Longrightarrow \mathbf{SCT_2}$ : Let $H \subseteq \Sigma^* \times \Sigma^*$ be the r.e. set satisfying (24) and (25) from Definition 9. We observe that every singleton $\{x\}$ is compact and $\{x\} \subseteq \bigcup \nu^{\rm fs}(u)$ iff $x \in \nu(u')$ for some $u' \ll u$. Let $H'$ be the set of all $(u', v')$ such that $u' \ll u$ and $v' \ll v$ for some $(u, v) \in H$. The $H'$ is r.e. and (22) and (23) are true for $H'$.

$\mathbf{SCT_2^{cc}} \Longrightarrow \mathbf{CT_2^{cc}}$ : A $\kappa$-name of a compact set $K$ is a list of all $u \in \Sigma^*$ such that $K \subseteq \bigcup \nu^{\mathrm{fs}}(u)$. Let $H \subseteq \Sigma^* \times \Sigma^*$ be the r.e. set satisfying (24) and (25). There is a machine $M$ that on input $(p,q)$ searches for $u, v \in \Sigma^*$ such that $u \ll p$, $v \ll q$ and $(u,v) \in H$. Then the function $f_M$ realizes the function $t^{\mathrm{cc}}$.

$\mathbf{CT_2^{cc}} \Longrightarrow \mathbf{CT_2^{pc}}$ : The function $x \mapsto \{x\}$ is $(\delta,\kappa)$-computable and the multi-function $(x,U) \rightrightarrows V$ mapping every $x \in X$ and $U \in \mathrm{range}(\bigcup \nu^{\mathrm{fs}})$ such that $x \in U$ to some $V \in \beta$ such that $x \in V$ is $(\delta, \bigcup \nu^{\mathrm{fs}}, \nu)$-computable. The multi-function $t^{\mathrm{pc}}$ is obtained from $t^{\mathrm{cc}}$ by composition, hence it is computable.

$\mathbf{CT_2^{pc}} \Longrightarrow \mathbf{CT_2}$ : By the same argument as above.

2. Suppose, there is a machine $M$ such that the function $f_M$ realizes the function $t^{\mathrm{pc}}$ from Definition 9. For $i \in \mathbb{N}$ let $p_i := \iota(0^i1)\iota(0^i1)\ldots$ and let $q_i$ be a list of all $w \in \mathrm{dom}(\bigcup \nu^{\mathrm{fs}})$ such that $0^i2 \ll w$. Then for all $i \notin A$, $\delta(p_i) = a_i$ and $\kappa(q_i) = \{b_i\}$, hence $f_M(p_i, q_i) = \langle 0^i1, v_i\rangle$ for some $v_i$ such that $0^i2 \ll v_i$ (such that $\{b_i\} \subseteq \bigcup \nu^{\mathrm{fs}}(v_i)$). Let $C$ be the set of all $i \in \mathbb{N}$ such that $f_M(p_i,q_i) = \langle 0^i1, v_i\rangle$ for some $v_i$ such that $0^i2 \ll v_i$. Since $C$ is r.e. and $A^c \subseteq C$ there is some $k \in C \cap A$. Let $t$ be the number of steps the machine $M$ operates on input $(p_k, q_k)$ until it halts. Let $w$ be the prefix of $q_i$ of length $t$. There is some $q' \in \Sigma^\omega$ such that $\kappa(wq') = \emptyset$. Also on input $(p_k, wq')$ the machine will halt in $t$ steps after writing $\langle 0^k1, v_k\rangle$ such that $0^k2 \ll v_k$, Since $k \in A$, $\{a_k\} = \nu(0^k2) \subseteq \bigcup \nu^{\mathrm{fs}}(v_k)$. But $\nu(0^k1) \cap \bigcup \nu^{\mathrm{fs}}(v_k) = \{a_k\} \cap \bigcup \nu^{\mathrm{fs}}(v_k)$ should be empty, since $\{a_k\} = \delta(p_k) \notin \kappa(wq')$. Contradiction. Therefore, the space $\mathbf{X}$ is not $CT_2^{\mathrm{pc}}$. □

In [Xu and Grubba 2009] $\mathrm{SCT}_2 \Longrightarrow (\mathrm{CT}_2^{\mathrm{cc}} \wedge \mathrm{CT}_2^{\mathrm{pc}} \wedge \mathrm{CT}_2)$ has been proved under the (unnecessary) assumption $U \neq \emptyset$ for all $U \in \beta$. We do not know whether the two remaining implications $\mathrm{SCT}_2 \Longrightarrow \mathrm{CT}_2^{\mathrm{cc}}$ and $\mathrm{CT}_2^{\mathrm{cc}} \Longrightarrow \mathrm{CT}_2^{\mathrm{pc}}$ are proper.

Axioms of separation are studied also in Intuitionistic Analysis [Troelstra 1966] and Constructive Analysis [Bishop and Bridges 1985]. In [Waaldijk 1996, Page 50] a topological space is called *effective* iff

$$(\forall x \in X)(\forall U,\ x \in U)(\forall y \in X)[y \in U \vee (\exists V)(x \in V \wedge y \notin V)].$$

In our framework this axiom corresponds to:

**Definition 12.**

IT: The multi-function $t$ mapping every $x, y \in X$ and $U \in \beta$ such that $x \in U$ to $(1,U)$ or to $(2,V)$ for some $V \in \beta$ such that $y \in U$ if the result is $(1,U)$ and $(x \in V \wedge y \notin V)$ if the result is $(2,V)$ is computable (more precisely, $(\delta,\delta,\nu,[\nu_{\mathbb{N}},\nu])$-computable).

**Theorem 13.** IT $\Longleftrightarrow$ $\mathrm{SCT}_2$

The proof is given in the next section.

## 7 Subspaces and Product Spaces

For a computable topological space $\mathbf{X} = (X, \tau, \beta, \nu)$ and $B \subseteq X$ the subspace $\mathbf{X}_B = (B, \tau_B, \beta_B, \nu_B)$ of $\mathbf{X}$ to $B$ is the computable topological space defined by $\mathrm{dom}(\nu_B) := \mathrm{dom}(\nu)$, $\nu_B(w) := \nu(w) \cap B$, see [Weihrauch and Grubba 2009, Section 8]. The separation axioms from Definitions 2 and 3 are invariant under restriction to subspaces.

**Theorem 14.** *If a computable topological space satisfies some separation axiom from Definitions 2, 3 and 9, then each subspace satisfies this axiom.*

**Proof:**

$\mathbf{CT_0}$ : Suppose, there is a computable function $f :\subseteq \Sigma^\omega \times \Sigma^\omega \to \Sigma^*$ that maps every pair $(p, q) \in \mathrm{dom}(\delta) \times \mathrm{dom}(\delta)$ such that $\delta(p) \neq \delta(q)$ to some $u$ such that $(\delta(p) \in \nu(u) \wedge \delta(q) \notin \nu(u))$ or $(\delta(p) \notin \nu(u) \wedge \delta(q) \in \nu(u))$. Suppose, $(p, q) \in \mathrm{dom}(\delta_B) \times \mathrm{dom}(\delta_B)$. Since $\delta(r) = \delta_B(r)$ for all $r \in \mathrm{dom}(\delta_B)$, $f(p, q) = U$ such that $(\delta(p) \in \nu(u) \wedge \delta(q) \notin \nu(u))$ or $(\delta(p) \notin \nu(u) \wedge \delta(q) \in \nu(u))$. Since $\nu_B(w) := \nu(w) \cap B$, $(\delta_B(p) \in \nu_B(u) \wedge \delta_B(q) \notin \nu_B(u))$ or $(\delta_B(p) \notin \nu_B(u) \wedge \delta_B(q) \in \nu_B(u))$.

$\mathbf{SCT_0}, \mathbf{CT_1}, \mathbf{CT_2}$ : Similar to $\mathbf{CT_0}$.

$\mathbf{WCT_0}$ : Suppose, the r.e. set $H$ satisfies (2) and (3) for the space $\mathbf{X}$. By (2), for all $x, y \in B$ there is some $(u, v) \in H$ such that $x \in \nu(u)$ and $y \in \nu(v)$, hence $x \in \nu_B(u)$ and $y \in \nu_B(v)$. Therefore, (2) is true for $\mathbf{X}_B$. Suppose, $\nu_B(u) \cap \nu_B(v) \neq \emptyset$. then $\nu(u) \cap \nu(v) \cap B \neq \emptyset$. By (3) there is some $x \in X$ such that $\nu(u) = \{x\} \subseteq \nu(v)$ or some $y \in X$ such that $\nu(v) = \{y\} \subseteq \nu(u)$. In the first case, if $x \notin B$ then $\nu_B(u) = \nu(u) \cap B = \emptyset$ (contradiction), hence $x \in B$ and $\nu_B(u) = \{x\} \subseteq \nu_B(v)$. Correspondingly, in the second case $y \in B$ and $\nu_B(v) = \{y\} \subseteq \nu_B(u)$. Therefore, (3) is true for $\mathbf{X}_B$.

$\mathbf{CT_0'}, \mathbf{CT_1'}, \mathbf{CT_2'}, \mathbf{SCT_2}$ : Similar to $\mathbf{WCT_0}$.

$\mathbf{CT_2^{pc}}$, $\mathbf{CT_2^{cc}}$ : This follows from [Weihrauch and Grubba 2009, Lemma 26] and the fact that $K \subseteq B$ is compact in $\mathbf{X}_B$ iff it is compact in $\mathbf{X}$. □

The product of two $T_i$-spaces is a $T_i$-space for $i = 0, 1, 2$. This is no longer true for some of the computable separation axioms. The product $\mathbf{X_1} \times \mathbf{X}_2 = \overline{\mathbf{X}} = (X_1 \times X_2, \overline{\tau}, \overline{\beta}, \overline{\nu})$ of two *computable* topological spaces $\mathbf{X}_1 = (X_1, \tau_1, \beta_1, \nu_1)$ and $\mathbf{X}_2 = (X_2, \tau_2, \beta_2, \nu_2)$, is defined by $\overline{\nu}\langle u_1, u_2\rangle = \nu_1(u_1) \times \nu_2(u_2)$ [Weihrauch and Grubba 2009, Section 8]. Let $\mathbf{R}$ be the computable real line from Example 1.

**Theorem 15.**

1. *The $SCT_2$-spaces are closed under product.*
2. *If $\mathbf{X_1} \times \mathbf{X_2}$ is $SCT_2$ and $X_2$ has a computable point, then $\mathrm{X}_1$ is $SCT_2$.*

3. *$\mathbf{X} \times \mathbf{R}$ is $WCT_0$ iff $\mathbf{X}$ is $SCT_2$.*

4. *For every axiom* T *such that* $\mathrm{SCT}_2 \Longrightarrow \mathrm{T} \Longrightarrow \mathrm{WCT}_0$ *the following statements are equivalent:*
   – $\mathrm{T} \iff \mathrm{SCT}_2$,
   – *the $T$-spaces are closed under product,*
   – *$\mathbf{X} \times \mathbf{R}$ is a $T$-space for every $T$-space $\mathbf{X}$.*

5. *The $WCT_0$-, $CT_0$- $CT_1$- and $CT_2$-spaces are not closed under product.*

**Proof:** 1. Suppose, $\mathbf{X_1}$ and $\mathbf{X}_2$ are $SCT_2$. By Theorem 7, $x_i \neq y_i$ is $(\delta_i, \delta_i)$-r.e. for $i = 1, 2$, hence $(x_1, x_2) \neq (y_1, y_2)$ is $([\delta_1, \delta_2], [\delta_1, \delta_2])$-r.e., hence again by Theorem 7, $\mathbf{X_1} \times \mathbf{X_2}$ is $SCT_2$.

2. Let $z = \delta_2(p')$ for some computable $p' \in \Sigma^\omega$. By Theorem 7.2, on $X_1 \times X_2$ the relation $(x_1, x_2) \neq (y_1, y_2)$ is $([\delta_1, \delta_2], [\delta_1, \delta_2]$-r.e. Therefore, there is a machine $M$ that halts on input $(\langle p_1, p'\rangle, \langle q_1, p'\rangle)$ for $p_1, q_1 \in \mathrm{dom}(\delta_1)$ iff $\delta(p_1) \neq \delta(q_1)$. Since $p'$ is computable, there is a machine $N$ that halts on input $(p_1, q_1)$ iff $\delta_1(p_1) \neq \delta_1(q_1)$, hence $x \neq y$ is $(\delta_1, \delta_1)$-r.e. By Theorem 7.2, $\mathbf{X_1}$ must be $SCT_2$.

3. Suppose $\mathbf{X} \times \mathbf{R}$ is $WCT_0$. An open basis set of $\mathbf{X} \times \mathbf{R}$ has the form $U \times (a; b)$ with rational $a < b$. Therefore, no set $\{(x, y)\}$ for $(x, y) \in X \times R$ is open. By Corollary 6, $\mathbf{X} \times \mathbf{R}$ is $SCT_2$. By 2. of this theorem, $\mathbf{X}$ is $SCT_2$. Suppose $\mathbf{X}$ is $SCT_2$. Since $\mathbf{R}$ is $SCT_2$, $\mathbf{X} \times \mathbf{R}$ is $SCT_2$ hence $WCT_0$.

4. Suppose $\mathrm{T} \iff \mathrm{SCT}_2$. Then $T$-spaces are closed under product by 1. of this theorem. Then $\mathbf{X} \times \mathbf{R}$ is a $T$-space for every $T$-space $\mathbf{X}$, since $\mathbf{R}$ is an $SCT_2$-space and hence a $T$-space. Suppose, $\mathbf{X} \times \mathbf{R}$ is a $T$-space for every $T$-space $\mathbf{X}$. Let $\mathbf{Y}$ be a $T$-space. Then $\mathbf{Y} \times \mathbf{R}$ is a $T$-space, hence a $WCT_0$-space. By 3. of this theorem, Y is $SCT_0$.

5. This follows from 4. of this theorem and Theorems 5 and 8. □

Since we do not know whether $\mathrm{SCT}_0 \iff \mathrm{CT}_2^{\mathrm{cc}}$ or $\mathrm{SCT}_0 \iff \mathrm{CT}_2^{\mathrm{pc}}$, we do not know whether the $CT_2^{\mathrm{cc}}$-spaces and the $CT_2^{\mathrm{pc}}$-spaces are closed under product. Finally, we prove Theorem 13.

**Proof:** (Theorem 13) By Theorem 15.4, it suffices to prove $\mathrm{SCT}_2 \Longrightarrow \mathrm{IT} \Longrightarrow \mathrm{WCT}_0$ and that the $IT$-spaces are closed under product.

$\mathbf{SCT_2} \Longrightarrow \mathbf{IT}$: Let $H$ be the r.e. set from the Definition of $\mathrm{SCT}_2$ in Definition 3. There is a machine $M$ that on input $(p, q, u)$ tries to show $u \ll q$ and simultaneously tries to find some $(v, w) \in H$ such that $v \ll p$ and $w \ll q$. If $u \ll q$ has been shown it writes $\langle w_1, u\rangle$, and if $(v, w) \in H$ has been found it writes $\langle w_2, v\rangle$ (where $\nu_{\mathbb{N}}(w_1) = 1$ and $\nu_{\mathbb{N}}(w_2) = 2$). Suppose, $\delta(p) = x \in U = \nu(u)$ and $\delta(q) = y$. The machine halts on input $(p, q, u)$, since $u \ll q$ can be proved if $x = y$, and some $(v, w) \in H$ can be found if $x \neq y$. If the result is $\langle w_1, u\rangle$ then $u \ll q$ hence $y \in U$. If the result is $\langle w_2, v\rangle$ then there is some $w$ such that $(v, w) \in H$,

$x \in \nu(v)$ and $y \notin \nu(v)$. Therefore the machine realizes the multi-function $t$ from Definition 12.

**IT** $\Longrightarrow$ $\mathbf{CT_0}$ Since by our general assumption $\mathbf{X}$ is a $T_0$-space, for every $x, y \in X$ such that $x \neq y$ there is some $U \in \beta$ such that $x \in U$ and $y \notin U$ or $y \in U$ and $x \notin U$. There is a machine that on input $(x, y)$ applies the multi-function $t$ to $(x, y, U)$ and to $(y, x, U)$ in turn for all $U \in \beta$ until some $V \in \beta$ is found such that $((2, V) \in t(x, y, U)$ and $x \in V)$ or $((2, V) \in t(y, x, U)$ and $y \in V)$ and then gives $V$ as its result. This machine computes the multifunction $t_0$ from Definition 2.

**The** $IT$**-spaces are closed under product:** Let $\mathbf{X}_1 = (X_1, \tau_1, \beta_1, \nu_1)$ and $\mathbf{X}_2 = (X_2, \tau_2, \beta_2, \nu_2)$ be $IT$-spaces. Suppose $(x_1, x_2) \in U_1 \times U_2$ and let $(y_1, y_2) \in X_1 \times X_2$ be another point. For $i = 1, 2$ there is a machine that on input $(x_i, y_i, U_i)$ produces $(1, U_i)$ or $(2, V_i)$ such that $y_i \in U_i$ if the result is $(1, U_i)$ and $(x \in V_i \wedge y \notin V_i)$ if the result is $(2, V_i)$. Combining both machines we get a machine that on input $((x_1, x_2), (y_1, y_2), U_1 \times U_2)$ yields

$$\begin{array}{ll} (1, U_1 \times U_2) & \text{if the results are } (1, U_1) \text{ and } (1, U_2), \\ (2, V_1 \times U_2) & \text{if the results are } (2, V_1) \text{ and } (1, U_2), \\ (2, U_1 \times V_2) & \text{if the results are } (1, U_1) \text{ and } (2, V_2), \\ (2, V_1 \times V_2) & \text{if the results are } (2, V_1) \text{ and } (2, V_2). \end{array}$$

Obviously, this machine computes the multifunction $t$ from Definition 12 for $\mathbf{X_1} \times \mathbf{X_2}$. □

## 8 Final Remarks

There may be other interesting axioms T of computable separation between $\mathrm{WSCT}_0$ and $\mathrm{SCT}_2$. By Theorem 15 only the $SCT_2$-spaces are closed under product and, hence, are the most natural ones. We do not know whether the implications $\mathrm{CT}_2^{\mathrm{pc}} \Longrightarrow \mathrm{CT}_2^{\mathrm{cc}}$ and $\mathrm{CT}_2^{\mathrm{cc}} \Longrightarrow \mathrm{SCT}_2$ are proper. Several other axioms concerning compact sets instead of points have not been considered in this article, for example $\mathrm{CT}_1^{\mathrm{cp}}$: The multi-function mapping each *compact* set $K$ and each *point* $y$ such that $y \notin K$ to some open set $V$ such that $K \subseteq V$ and $y \notin V$ is $(\kappa, \delta, \bigcup \nu^{\mathrm{fs}})$-computable.

The computable topology developed here and in [Weihrauch and Grubba 2009] is pointless topology. The "concrete objects" are the names of base elements ($\nu :\subseteq \Sigma^* \to \beta$) which are considered as "frames" or "regions" that can be filled with points. Names of other objects are composed from names of base elements ($\delta$, $\theta$, $\kappa$ etc.) [Weihrauch and Grubba 2009, Definition 5, Section 10]. No axiom requires the existence of points, non-empty open sets etc., see Theorem 14.

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