Journal of Universal Computer Science, vol. 12, no. 2 (2006), 214-232
submitted: 4/10/04, accepted: 9/5/05, appeared: 28/2/06

A Structure Causality Relation for Liveness Characterisation in Petri Nets

Belhassen Zouari
(LIP2 laboratory , University ElManar -Tunisia
belhassen.zouari@fst.rnu.tn)

Abstract: Characterising liveness using a structure based approach is a key issue in theory of Petri nets. In this paper, we introduce a *structure causality relation* from which a topological characterisation of liveness in Petri nets is defined. This characterisation relies on a controllability property of siphons and allows to determine the borders of the largest abstract class of Petri nets for which equivalence between liveness and deadlock-freeness holds. Hence, interesting subclasses of P/T systems, for which membership can be easily determined, are presented. Moreover, this paper resumes, from a new point of view, similar results related to this issue and, provides a unified interpretation of the causes of the non-equivalence between liveness and deadlock-freeness.

Keywords: Petri nets, structural analysis, siphons, liveness, deadlock-freeness
Categories: F.4, F.3.1, F.1.1., D.0, G.2

1 Introduction

Place/Transition (P/T) nets [Reisig 91] are well-known models for the representation and analysis of concurrent systems. The use of structural methods for the behavioural analysis of such systems presents two major advantages with respect to other approaches: the state explosion problem inherent to concurrent systems is avoided, and the investigation of the relationship between the behaviour and the structure (the graph theoretical structures and linear algebraic properties associated with the net and its initial marking) usually leads to a deep understanding of the system. Here, we deal with liveness, i.e., the fact that every transition can be enabled again and again. It is well known that this behavioural property is as important as formally hard to treat. Although some structural techniques can be applied to general nets, the most satisfactory results are obtained when the interplay between conflict and synchronisation is limited. An important theoretical result is the concept of controlled siphon, i.e. siphon that cannot be *insufficiently* marked. Indeed, the controlled siphon property is a necessary liveness condition, and a sufficient deadlock-freeness condition. Although the mechanisms ensuring the control of siphons are not all recognised, it is important to obtain a deeper understanding of the causes of the non-equivalence between liveness and deadlock-freeness. The present work is part of this undertaking, concentrating on basic concepts and theoretical results. More precisely, it deals with a refined characterisation of the "topological construct" making possible the simultaneous existence of dead transitions and live transitions. The interest of such a characterisation is to make applicable the structural analysis techniques, in particular those related to controlled siphon property, for P/T systems where the interplay between conflict and synchronisation is relaxed. In [Barkaoui and Zouari 03], we presented notions of ordered transition and root as a basis of a non-liveness characterisation and related class definitions. In this paper, we notably revisit this last providing related proper formulations. The paper is organised as follows. In section 2 the basic concepts and notations on P/T systems are recalled. Section 3 is devoted to the exploration of some consequences of the deadlock freeness property under non-liveness hypothesis. The "topological construct" deduced from this exploration not only extends the class of systems for which equivalence between liveness and deadlock-freeness is ensured, but also permits to revisit some known structural theory results from a new point of view. In section 4, we introduce the concept of death causality relation, then we present a large abstract class of P/T systems and some related subclasses, for which controlled siphon property is a necessary and sufficient liveness condition. We compare our results with some similar works, then we present, in section 5, computational methods based on structural properties defined in section 4. Finally, we conclude with a summary of our results.

2 Basic Definitions and Notations

This section contains the basic definitions and notations of Petri nets theory [Reisig 91] which will be needed in the remainder of the paper.

2.1 Place/Transition systems

Definition 2.1

A P/T net is a tuple N = <P, T, F, V> where :

(i) $P \neq \emptyset$ is a finite set of node *places* ;

(ii) $T \neq \emptyset$ is a finite set of node *transitions* ;

(iii) $F \subseteq (P\times T) \cup (T\times P)$ is the *flow relation* ;

(iv) $V : F \rightarrow \mathbb{N}^+$ is the *weight function* (valuation) ;

N is a weighted bipartite graph (where $P\cap T=\emptyset$).

In the following, we consider a P/T net N= < P, T, F, V >. For technical reasons and in order to avoid to treat some uninteresting particular nets, we rule out isolated places.

Definition 2.2

- The *preset* of a node $x \in P \cup T$ is defined as ${}^\bullet x = \{y \in P \cup T \,/\, (y, x) \in F\}$.
- The *postset* of $x \in P \cup T$ is defined as $x^\bullet = \{y \in P \cup T \,/\, (x, y) \in F\}$.
- The preset (postset) of a set is the union of the presets (postsets) of the elements.
- The *subnet* induced by P' with $P' \subseteq P$ is the net N' = < P', T', F', V' > where $T' = {}^\bullet(P') \cup (P')^\bullet$; $F' = F \cap ((P'\times T') \cup (T'\times P'))$ and V' is the restriction of V on F'.

The subnet induced by T' with $T' \subseteq T$ is defined analogously.

Definition 2.3

- A shared place p ($|p^\bullet| \geq 2$) is said to be *homogeneous* iff
  $\forall t, t' \in p^\bullet$, V(p, t)= V(p, t'), (V(p,t) is also denoted by V(p))

  If all shared places of P are homogeneous: the valuation V is said to be homogeneous.
- A place $p \in P$ is said to be *non-blocking* iff
  $$p^\bullet \neq \varnothing \Rightarrow \text{Min}_{t \in {}^\bullet p} \{V(t,p)\} \geq \text{Min}_{t \in p^\bullet} \{V(p,t)\}.$$
- The definition of the valuation function V of a P/T net N can be extended to : W: $(P\times T) \cup (T\times P) \rightarrow \mathbb{N}$ such that :
  $$W(u) = V(u) \text{ if } u \in F \text{ ; } W(u) = 0 \text{ otherwise.}$$

Example

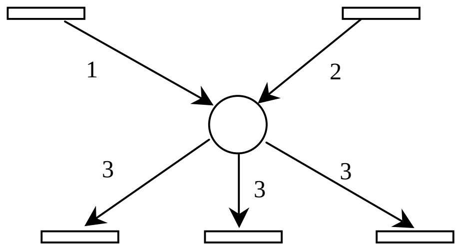

Figure 1 : a subnet with homogenous valuation

Definition 2.4

- The matrix C indexed by $P \times T$ defined by C(p, t) = W(t, p) - W(p, t) is called the incidence matrix of the net.
- An integer vector f ($f \neq 0$) indexed by P ($f \in \mathbf{Z}^P$) is a *P-invariant* iff $f^t.C=0^t$.
- An integer vector g ($g \neq 0$) indexed by T ($g \in \mathbf{Z}^T$) is a *T- invariant* iff C.g =0
- $\| f \| = \{p \in P \mid f(p) \neq 0\}$ ($\| g \| = \{t \in T : g(t) \neq 0\}$) is called the *support* of f (g).

We denote by $\| f \|^+ = \{p \in P : f(p) > 0\}$ and by $\| f \|^- = \{p \in P : f(p) < 0\}$.

Definition 2.5

- A *marking* M of a P/T net N = < P, T, F, V > is a mapping $M : P \rightarrow \mathbb{N}$,
  M(p) denotes the number of tokens contained in place p.
- The pair <N, M_0> is called a ***P/T system*** and M_0 is called the initial marking.
- The transition $t \in T$ is called *enabled* under M, in symbols $M [t\rangle$, iff
  $$\forall p \in {}^\bullet t: M(p) \geq V(p,t).$$
- If $M[t\rangle$, the transition t may *occur (fire)*, resulting in a new marking M', in symbols $M[t\rangle M'$, with $\forall p \in P,\ M'(p) = M(p) - W(p, t) + W(t, p)$.

- The set of all reachable markings, in symbols $Acc(M_0)$, is the smallest set such that : $M_0 \in Acc(M_0)$; $M \in Acc(M_0)$ and $M[t\rangle M' \Rightarrow M' \in Acc(M_0)$. If $M_0\ [t_1\rangle M_1[t_2\rangle \ldots M_{n-1}[t_n\rangle M_n$, then $\sigma = t_1 t_2 \ldots t_n$ is called an *occurrence sequence (firing sequence).*

In the following, we recall the definition of some basic behavioural properties.

Definition 2.6. Let $<N, M_0>$ be a P/T system.

- A transition $t \in T$ is said to be *dead* for $M^* \in Acc(M_0)$ iff $\nexists\ M \in Acc(M^*)$ where $M[t\rangle$.
- A marking $M \in Acc(M_0)$ is called a *dead marking* iff $\forall t \in T$, t is dead for M.
- $<N, M_0>$ is *deadlock-free* (no dead marking) iff $\forall M \in Acc(M_0), \exists t \in T: M\ [t\rangle$.
- A transition $t \in T$ is said to be *live* for M_0 iff $\forall M \in Acc(M_0), \exists M' \in Acc(M): M'[t\rangle$.
- $<N, M_0>$ is *live* iff $\forall t \in T$, t is live for M_0 (is not live iff $\exists t \in T$ dead for $M \in Acc(M_0)$).
- A place $p \in P$ is said to be *sufficiently marked* for $M \in Acc(M_0)$ iff $M(p) \geq \text{Min}_{t \in p^\bullet} \{V(p,t)\}$.
- A place $p \in P$ is said to be *bounded* for M_0 iff $\exists k \in \mathbb{N}, \forall M \in Acc(M_0)$: $M(p) \leq k$.
- $<N, M_0>$ is *bounded* iff $\forall p \in P$: p is bounded for M_0.

Typically, many subclasses are defined by restricting/eliminating the interleaving between choices and synchronisations. Among them:

Definition 2.7. *P/T net subclasses*

- State machines (SM) are P/T nets where each transition has one input place and one output place, i.e., $\forall t \quad |{}^\bullet t| = |t^\bullet| = 1$.
- Marked Graphs (MG) are P/T nets where each place has one input and one output transition, i.e., $\forall p \quad |{}^\bullet p| = |p^\bullet| = 1$.
- Join free (JF) nets are P/T nets in which each transition has at most one input place, i.e., $\forall t \in T, |{}^\bullet t| \leq 1$.
- Choice free (CF) nets [Teruel et al., 97] are P/T nets in which each place has at most one output transition, i.e., $\forall p, |p^\bullet| \leq 1$.
- Free choice (FC) nets [Desel et al., 95] are P/T nets in which conflicts are always equal, i.e., $\forall t,t'$, if ${}^\bullet t \cap {}^\bullet t' \neq \emptyset$, then ${}^\bullet t = {}^\bullet t'$.
- Equal Conflict (EC) nets [Teruel et al., 96] are the weighted generalisation of free choice nets, i.e., if ${}^\bullet t \cap {}^\bullet t' \neq \emptyset$, then $\forall p, W(p, t) = W(p, t')$.

- Asymmetric choice (AC) nets [Barkaoui et al., 95] are P/T nets in which conflicts are ordered, i.e., $\forall p,q \in P$, if $p^\bullet \cap q^\bullet \neq \emptyset$, then $p^\bullet \subseteq q^\bullet$ or $q^\bullet \subseteq p^\bullet$.

2.2 Controlled Siphon property

A key notion of structure theory is the siphon substructure. Let $<N, M_0>$ be a P/T system

Definition 2.8 A nonempty set $S \subseteq P$ is called a *siphon* iff ${}^\bullet S \subseteq S^\bullet$.
S is *minimal* iff it contains no other siphon as a proper subset.

In the following, we assume that all P/T nets have *homogeneous valuation*.

Definition 2.9 Let S be a siphon of $<N, M_0>$,
S is said to be *controlled* iff S is sufficiently marked at any reachable marking :

$$\forall M \in Acc(M_0), \exists p \in S: M(p) \geq V(p)$$

Definition 2.10 $<N, M_0>$ is said to be satisfying the *controlled-siphon property (cs-property)* iff all minimal siphons of $<N, M_0>$ are controlled.

We recall two well-known basic relations (easy to prove) between liveness and cs-property [Barkaoui et al., 95]. The first states that the cs-property is a sufficient deadlock-freeness condition, the second states that the cs-property is a necessary liveness condition.

Proposition 2.11
$<N, M_0>$ satisfies the cs-property $\Rightarrow$ $<N, M_0>$ is deadlock-free (weakly live).

Proposition 2.12 $<N, M_0>$ is live $\Rightarrow$ $<N, M_0>$ satisfies the cs-property.

In order to check the cs-property, two main structural conditions (*sufficient but not necessary*) permitting to determine whether a given siphon is controlled are developed in [Barkaoui et al., 95] [Lautenbach et al., 94].

Proposition 2.13 Let S be a siphon of $<N, M_0>$. If one of the two following conditions holds, then S is controlled.

1. $\exists R \subseteq S$ such that : $R^\bullet \subseteq {}^\bullet R$, R is sufficiently marked at M_0, and all places of R are non-blocking (S is said to be trap controlled)
2. $\exists$ a *P-invariant* f ($f \in \mathbf{Z}^P$) such that $S \subseteq \| f \|$ and $\forall p \in (\| f \|^- \cap S) : V(p) = 1$, $\| f \|^+ \subseteq S$, and $\sum_{p\in P} [f(p) . M_0(p)] > \sum_{p\in S} [f(p). (V(p)-1)]$.

   (S is said to be invariant controlled)

If cs-property is a sufficient liveness condition, then the equivalence between liveness and deadlock freeness holds. In [Barkaoui et al., 95], this equivalence is proven for the class of (unbounded) homogeneous Asymmetric Choice (AC) systems. This class contains Join free systems (JF), Equal Conflict (EC) systems [Teruel et al., 96], and Free Choice (FC) nets. Let us recall that for FC nets, the cs-property coincides with the well-known Commoner's property where liveness is monotonic [Barkaoui et al., 95] (since siphons are all trap-controlled). Moreover, for bounded FC nets, the liveness property is decidable in polynomial time [Desel, 92] [Esparza et al., 92].

3 A Non-Liveness Structural Characterisation

In the following, we deal with P/T systems with homogeneous valuation and satisfying the cs-property. Our goal is to refine the "topological construct" behind the “pathological” behaviour characterised by a simultaneous existence of dead and live transitions (deadlock-free but not live).

First, let us present a general statement. If we consider a P/T system which is deadlock-free but not live, then one can find a singular reachable marking M* from which we can partition the set T of transitions into a subset T_D of dead transitions and a subset T_L of live transitions.

Proposition 3.1 *Singular marking M**

Let $<N, M_0>$ be a P/T system which is *deadlock-free* but *not live* and T the set of transitions. There exists a marking $M^* \in Acc(M_0)$ such that

$T = T_D \cup T_L$; $T_D \cap T_L = \emptyset$; $T_D \neq \emptyset$, $T_L \neq \emptyset$;

where $\forall t \in T_L$, t is live for M* ,

and $\forall t' \in T_D$, t' is dead for M* (i.e. not live for M_0)

M* is called a singular marking of $<N, M_0>$ and (T_L, T_D) the associated partition.

Proof: easy to prove by stating that the set of dead transitions is monotonously increasing while transitions occur ; therefore, it finally can reach a maximum. The marking reached is singular. As $<N, M_0>$ is deadlock-free, $T_L \neq \emptyset$ and, as it is not live, $T_D \neq \emptyset$.

<u>Remark</u> : the singular marking M* (and its associated partition (T_L, T_D)) may not be unique for a given P/T system, but there exists at least one. It is important to note that T_D is maximal in the sense that all transitions that do not belong to T_D, will never become dead.

We are only interested in the existence of a singular marking and its associated partition, but not in how to determine it.

In order to define a non-liveness characterisation, let us introduce some new structural concepts based on *causality* relations between *dead transitions*. Hence, we define an *ordered transition* as a transition for which all its *input places* are *comparable*. By comparable places, we mean that we can apply the *inclusion relation* on the transition sets associated with their output arcs.

Definition 3.2 *Ordered transitions*

Let N = < P, T, F, V > be a P/T net . $t \in T$ is said to be *ordered* iff

$$\forall p, q \in {}^\bullet t \ : \ p^\bullet \subseteq q^\bullet \text{ or } q^\bullet \subseteq p^\bullet \ .$$

It is worth to note that the condition to being ordered is the property characterising AC nets.

Moreover, we introduce the concept of *root*, associated with a transition, as an input place that is *minimum* in the sense of the inclusion relation seen above. As the minimum may not be unique, we define the set of roots associated with a transition.

Definition 3.3 *Root of a transition*

Let N = < P, T, F, V > be a P/T net . Let $t \in T$ / ${}^\bullet t \neq \emptyset$.

$r \in {}^\bullet t$ is called a *root place* of t iff $\forall p \in {}^\bullet t, \ r^\bullet \subseteq p^\bullet$

$Root(t) = \{r \in {}^\bullet t \mid r^\bullet \subseteq p^\bullet, \ \forall p \in {}^\bullet t\}$ is called the *root* of t

One can remark that : ${}^\bullet t \neq \emptyset$, Root (t) = $\emptyset \Rightarrow |{}^\bullet t| \geq 2$, and

Root (t) $\neq \emptyset \Rightarrow |\text{Root}(t)| \leq |{}^\bullet t|$.

$\forall r,s \in$ Root(t), $r^\bullet = s^\bullet$.

An ordered transition *admits* obviously a root place (i.e. Root(t) $\neq \emptyset$), but a transition admitting a root place is not necessarily ordered.

<u>Example</u>

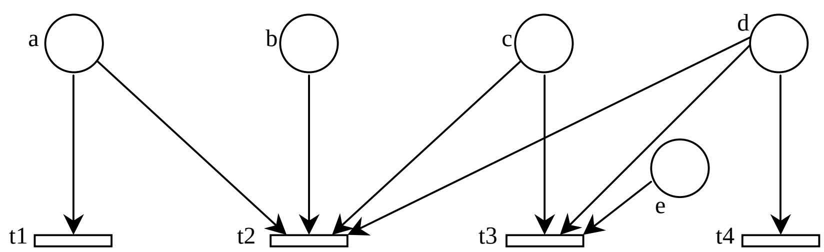

Figure 2 : subnet with ordered and non-ordered transitions
Root (t1) = {a}, Root (t2) = {b}, Root (t3) = {e}, Root (t4) = {d}.
transitions t1, t3 and t4 are ordered ; transition t2 is not ordered

It is worth to note that the class of AC nets can be defined as nets where all transitions are ordered.

Let us now establish a non-liveness characterisation on the basis of the cs-property and using the concept of non-ordered transitions.

Theorem 3.4

Let <N, M_0> be a *not live* P/T system satisfying the *cs-property*, and M* be a singular marking of <N, M_0>. There exists a *non-ordered dead* transition t* for M*.

Before proving Theorem 3.4, we establish a lemma that defines a particular partition of the preset of transition t* (i.e. the set of its input places), for which some important properties are exhibited.

Lemma 3.5

Let <N, M_0> be a *not live* P/T system satisfying the *cs-property*, M* be a singular marking of <N, M_0> and t* a *dead* transition for M*. Then, the input transitions set $^\bullet(t^*)$ can be partitioned in three subsets $^\bullet(t^*) = D_P(t^*) \cup LD_P(t^*) \cup LL_P(t^*)$ such that :

1/ $D_P(t^*) = \{p \in {}^\bullet(t^*) \mid {}^\bullet p \cap T_L = \varnothing\}$ (places with *Dead* input transitions), verifies : $\forall p \in D_P(t^*)$, $\forall M \in Acc(M^*) : M(p) \geq V(p,t^*)$ and $M(p) = M^*(p)$

2/ $LD_P(t^*) = \{p \in {}^\bullet(t^*) \mid {}^\bullet p \cap T_L \neq \varnothing$ and $p^\bullet \cap T_L = \varnothing\}$ (places with *Live* input transitions and *Dead* output transitions), verifies : $\forall p \in LD_P(t^*), \exists M \in Acc(M^*) : M(p) \geq V(p,t^*)$

3/ $LL_P(t^*) = \{p \in {}^\bullet(t^*) \mid {}^\bullet p \cap T_L \neq \varnothing$ and $p^\bullet \cap T_L \neq \varnothing\}$ (places with *Live* input transitions and *Live* output transitions). $LL_P(t^*)$ contains at least two items and verifies : $\forall M \in Acc(M^*), \exists p \in LL_P(t^*) : M(p) < V(p,t^*)$

Intuitively, $D_P(t^*)$ is the set of input places of t* that will never be "supplied" by tokens (i.e. their input transitions are dead for M*). These places will always remain sufficiently marked. $LD_P(t^*)$ is the set of input places of t* that would be regularly "supplied" but never "emptied" ; these places would be sufficiently marked and are not bounded. $LL_P(t^*)$ is the set of input places of t* that would be regularly "supplied" and regularly "emptied", but are never simultaneously sufficiently marked. The places of $LL_P(t^*)$ are at the origin of the non-liveness of t*.

Proof of Lemma 3.5 :

In order to prove the first point, we have to prove that :

$$\forall p \in {}^\bullet(t^*) : {}^\bullet p \cap T_L = \varnothing \Rightarrow p^\bullet \cap T_L = \varnothing \qquad \text{(A1)}$$

Suppose (A1) is not true. In this case, there exists a place p with all its input transitions in T_D (i.e. ${}^\bullet p \cap T_L = \varnothing$) and at least one output transition t_v in T_L (i.e. $p^\bullet \cap T_L \neq \varnothing$). Since t_v is live, there exists a firing sequence that makes place p not sufficiently marked (as we deal with homogeneous valuations : M(p)<V(p)) because all its input transitions are dead. So, t_v becomes dead and contradicts the maximality of T_D.

Suppose now that the first point of the Lemma is false. Since M*∈ Acc(M*), we have $\forall t \in T_D$, $\exists p_t \in {}^\bullet t : {}^\bullet p_t \cap T_L = \emptyset$ and $M^*(p_t) < V(p_t,t)$. Let $S=\{p_t\}_{t\in T_D}$. By construction, ${}^\bullet S \subseteq T_D$ and $T_D \subseteq S^\bullet$ (for all p_t in S, ${}^\bullet p_t \cap T_L = \emptyset$). So, S is a siphon. Since $\forall p_t \in S$, $M^*(p_t) < V(p_t,t)$, S is not sufficiently marked for M* and hence, the cs-property is contradicted. If S is not minimal, then there is a minimal siphon included in S which is not sufficiently marked for M* : this also contradicts the cs-property. Using now the assertion A1 (if p has no live input then all outputs of p are dead), we can deduce that the marking of such places remains unchanged for all markings reachable from M*, and point 1 is proved.

The second point of the lemma can be easily proved : as places in $LD_P(t^*)$ have live input transitions, there exists a firing sequence from M* that makes any of these places sufficiently marked. As all their output transitions are dead, there exists a firing sequence from M* that makes all these places sufficiently marked and remain sufficiently marked for ever. One can remark that places of $LD_P(t^*)$ are not bounded.

To prove the third point of the lemma, we first prove that $LL_P(t^*)$ is not empty ($Card(LL_P(t^*))>0$). Suppose that $LL_P(t^*) = \emptyset$: any input place of t* having a live input transition (there is at least one because ${}^\bullet({}^\bullet t^*)\cap T_L \neq \emptyset$) has no live output transition. As the other input places of t* (belonging to $D_P(t^*)$ or $LD_P(t^*)$) are such that their pre-conditions on t* are satisfied at a given M**∈ Acc(M*) and remain satisfied (point 1 or point 2 of lemma), we can reach a marking M from M** such that t* would be enabled at M. So, $Card(LL_P) \geq 1$. Suppose now that $Card(LL_P(t^*))=1$ ($LL_P(t^*) =\{p1\}$). As p1 has a live input transition and as we deal with homogeneous valuations, we would reach a marking M from M* such that p1 is sufficiently marked for M. Since the other pre-conditions of t* are satisfied for a given M**∈ Acc(M*) and remain satisfied (point 1 or point 2 of lemma), then t* would be enabled at this marking, which contradicts t* is dead. So, $Card(LL_P(t^*)) \geq 2$. The last property of point 3 of this lemma is easy to prove : if it is not true, then there exists a marking M**∈ Acc(M*) where all places of $LL_P(t^*)$ are sufficiently marked, and where all other input places of t* are sufficiently marked, that enables t* and contradicts the hypothesis. □

Proof of Theorem 3.4 :

By Proposition 3.1, there exists a singular marking M* and a dead transition t* for M* ($t^*\in T_D$). We only have to prove that t* is non-ordered. From the previous lemma, one can deduce that there exists a marking M**∈ Acc(M*) satisfying $\forall p\in {}^\bullet(t^*) \setminus LL_P(t^*) : M^{**}(p) \geq V(p,t^*)$.

Suppose, by contradiction, that t* is *ordered*. Thus, the set of places ${}^\bullet(t^*)$ can be ordered. Hence, one can derive an ordering on $LL_P(t^*)$. Suppose $LL_P(t^*)=\{p_1,p_2, \ldots p_k\}$ and, without loss of generality, we may assume $p_1^\bullet \subseteq \ldots \subseteq p_k^\bullet$. As $p_1 \in LL_P(t^*)$ (i.e. $p_1^\bullet \cap T_L \neq \emptyset$), $\exists t\in p_1^\bullet$ such that t is enabled at a marking M reachable from M**. Since $\forall p \in LL_P(t^*)$: $p \in {}^\bullet t$ (because $p_1{}^\bullet \subseteq \ldots \subseteq p_k{}^\bullet$) and valuation is homogeneous, t* would be enabled at M, that contradicts the dead hypothesis of t*. Thus, t* is a *non-ordered* transition. □

Example

The following figure illustrates a P/T system for which Theorem 3.4 is applicable.

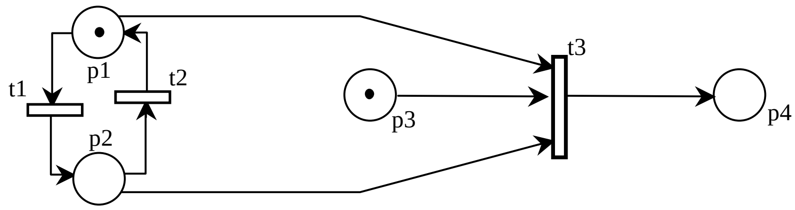

Figure 3 : a P/T system not live but satisfying the cs-property

S1={p1,p2} and S2={p3} are the minimal siphons of this system. S1 and S2 are controlled and therefore, the system satisfies the cs-property. Transition t3 is a non-ordered dead transition for the present marking. One may remark that if p3 is not marked (i.e. M*(p3)=0), siphon S2 will not be controlled. Regarding the previous lemma, $D_P(t3)$={p3}, $LD_P(t3)=\emptyset$ and $LL_P(t3)$={p1, p2}.

As we consider *non-ordered transitions* as well as ordered ones, it is important to note, that Theorem 3.4 establishes a more general non-liveness characterisation than that of AC systems (where only ordered transitions are considered). Hence, one can derive, from Theorem 3.4, a unified proof that cs-property is a necessary and sufficient liveness condition for the particular systems where all transitions are ordered (AC systems).

We established a non-liveness characterisation based on the concept of *structural order* associated with transitions. If we only consider the case of ordered transitions (as it is the case of AC nets), the link between cs-property and liveness is easily identified. This non-liveness refined characterisation not only permits to revisit this known result [Barkaoui et al., 95] from a new perspective, but using a topological reasoning, it would also extend the applicability domain of the cs-property to systems with non-ordered transitions.

4 A Death Causality Relation and DC-Systems

In this section, we are concerned by P/T systems admitting non-ordered transitions which represent a more general class, as the interplay between conflict and synchronisation is more relaxed due to the presence of non-ordered transitions. For this general class, we aim to establish a necessary and sufficient liveness condition for P/T systems using the cs-property.

Let us consider P/T systems with non-ordered transitions. We denote by T_o the subset of ordered transitions and by T_{no} the subset of non-ordered transitions :

$$T = T_o \cup T_{no} \;;\; (T_o \cap T_{no} = \emptyset)$$

In a first stage, we define a new *causality relation* between transitions which relies on *death dependencies*. We call this relation the *death causality relation*. It is based on the following principle : a transition t' is causally dependent upon transition t, if the death of t *leads to* (implies) the death of t'. More precisely, transition t' becomes dead for a marking M*, once transition t is assumed to be dead for M*.

Definition 4.1 *death causality relation ($\rightarrow$)*

Let <N, M_0> be a P/T system with N= <P, T, F, V>, and $M^* \in Acc(M_0)$ a singular marking.

The death causality relation (denoted by $\rightarrow$) is defined on T as follows :

$$\forall\, t, t' \in T, \quad t \rightarrow t' \text{ if and only if } [\text{ t is dead for } M^* \Rightarrow \text{ t' is dead for } M^*]$$

The death causality relation can be determined using topological based properties of the P/T system as the death propagation by root (proposition 5.1), by pipe (proposition 5.2) or by bounded place (proposition 5.3). Now, let us define, for a given transition t, the set of *all* transitions that are causally dependent upon t according to the death causality relation previously defined. We call this set the *death causality set* associated with transition t.

Definition 4.2 *death causality set*

Let <N, M_0> be a P/T system and $M^* \in Acc(M_0)$ a singular marking. Let t be any transition of T.

The death causality set associated with t (denoted by D(t)) is the maximum set defined as follows : $D(t) = \{ t' \in T : \; t \rightarrow t' \}$

Hence, the set D(t) contains all transitions that become dead once transition t is assumed to be dead. We present further in this section a computation algorithm of D(t).

Let us now introduce a key property for the definition of DC-systems : if, for each non-ordered transition t, its associated death causality set D(t) coincides with the set T of all transitions, then the P/T system is said to be satisfying the DC-property (DC for "*Death Causality*", as D(t) covers the whole set of transitions for every non-ordered transition). Thus, we can present the class of DC-systems.

Definition 4.3 *Class of DC-systems*

Let <N, M_0> be a P/T system, T be the set of transitions and T_{no} be the subset of non-ordered transitions.

<N, M_0> is a *DC-system* if and only if $\forall t \in T_{no}$, D (t) = T .

Now, let's introduce a relevant result for the class DC-system that concerns a necessary and sufficient condition for liveness.

Theorem 4.4

Let <N, M_0> be a DC-system.

<N, M_0> is live if and only if <N, M_0> satisfies the controlled-siphon property

***Proof*:** the proof mainly relies on the non-liveness characterisation of Theorem 3.4

$\Rightarrow$) trivial

$\Leftarrow$) suppose <N, M_0> is a not live DC-system satisfying the cs-property by Theorem 3.4, <N, M_0> admits a non-ordered dead transition t*.

by definition of DC-system, D(t*)=T. Then, T_D=T (all transitions are dead) and cs-property is not satisfied. We obtain a contradiction.

From our better understanding of which requirements are at the heart of non-equivalence between deadlock-freeness and liveness, we define some interesting subclasses of DC-systems for which the membership issue is reduced to examining the net without requiring any exploration of the behaviour, i.e. for which membership can be structurally ensured.

First, we define a subclass of DC-systems called *Root net systems,* exploiting the causality relations between output and input transitions of bounded root places.

Definition 4.5 *Root net system*

Let <N, M_0> be a P/T system where N = < P, T, F, V > is a P/T net.

<N, M_0> is a *Root net system* iff the three following conditions hold :

(i) $\forall t \in T$: t admits a root place (i.e. Root(t) $\neq \emptyset$),

(ii) all root places are bounded.

(iii) The subnet N* induced by root places is strongly connected.

Determining whether a P/T system is a Root net system is easy from the computational point of view. Indeed, this task is similar to the computation of the local 'minimum', whenever it exists, to each transition among its input places (which minimum may not be unique).

It is worth to note that the transitions of a Root net system are not necessarily ordered, but requires that every transition has (at least) a root place.

Theorem 4.6

Let <N, M_0> be a Root net system.

<N, M_0> is live if and only if <N, M_0> satisfies the *controlled-siphon* property.

Proof

Due to the particular structure of N* (strong connectivity of the subnet induced by root places), one can ensure that $\forall t \in T_{no}$: D(t) = T in N* and a fortiori in N. Then cs-property implies liveness.

We define another interesting subclass of DC-systems, called *well-structured systems* (WS-systems), exploiting the fact that in every infinite occurrence sequence there must be a repetition of markings under boundedness hypothesis. Nets of this subclass are bounded and satisfy the following structural condition: we cannot get a T-invariant g such that T_{no} is not included in $||g||$. This class contains strictly the one T-invariant nets from which (ordinary) bounded nets covered by T-invariants can be approximated as proved in [Lautenbach et al., 94].

Definition 4.7 *Well-structured system*

Let <N, M_0> be a non-ordered system. <N, M_0> is a *well-structured system* (WS-system) if the two following conditions are satisfied :

(i) N is bounded.

(ii) $\forall$ T-invariant g, $T_{no} \subseteq \| g \|$

Theorem 4.8

Let <N, M_0> be a well-structured system.

<N, M_0> is live if and only if it satisfies the *controlled-siphon* property.

***Proof*:** Let <N, M_0> be satisfying the cs-property but not live. Consider a singular marking M* and its associated partition (T_L, T_D). Consider the subnet induced by T_L. According to Theorem 3.4, this subnet is live and bounded for M*. Hence, There exists necessarily a firing sequence for which count-vector is a T-invariant and not covering a non-ordered transition t*. This contradicts condition (ii) of definition 4.7.

Example of a DC-System :

The P/T system described in Figure 4 (where T_{no} ={t3}) satisfies the two previous conditions of well-structured system definition. Indeed, it is conservative (bounded for any M_0) and the non-ordered transition t3 is included in any T-invariant (moreover, by applying the structural rules, we can check that D(t3) = T).

This system is a DC-system. One can remark that it is neither an AC system nor a state machine decomposable. It contains four minimal siphons: S1={a,b,d}, S2= {e,c,f}, S3 ={e,b,d} and S4 = {a,f,d}.

This system is not an Extended Asymmetric Choice (EAC) system [Aalst et al., 98] because the acyclic relation required in the definition of EAC is not satisfied for input places (a and b) of transition t3.

Under the following four structural marking conditions, these siphons are invariant controlled: on S1: a +b+2d >0; on S2: e+c+f >1; on S3: a+b+d-f >0, on S4: a+f+d-e >0.

So, this DC-system is live for any marking M_0 satisfying the previous four conditions (e.g. M_0 = a +b + e +f).

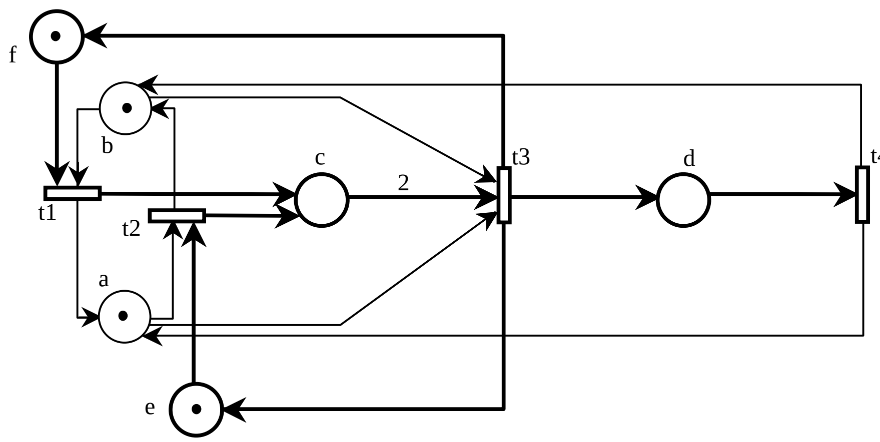

Figure 4 : Example of a DC-system

It is worth to note that the defined class of DC-systems contributes in providing a 'unified' understanding of the equivalence and the non-equivalence between liveness and deadlock-freeness in many known classes of nets.

Hence, one may observe, that a live and bounded AC system is a Root net system, but the converse is not true.

Extended Asymmetric Choice (EAC) systems presented in [Aalst et al., 98], permit test arcs and are defined on the basis of a non-cyclic relation on input places of any transition. Interesting structural characterisations have been exhibited and authors proved the Commoner's Theorem in one direction (necessary liveness condition) for EAC. By comparison, DC-systems introduce some "topological features", as non-ordered transition, that are not considered in EAC systems (see example of Figure 4), and for which a necessary and sufficient liveness condition is established using the cs-property.

Another interesting class of nets presenting similar results is multi-level deterministically synchronised processes (DS)*SP systems [Recalde et al., 01]. We recall that (DS)*SP were introduced to generalise the Deterministic Systems of Sequential Processes (DSSP) [Souissi, 93]. A (DS)*SP system is recursively built, starting from state machine modules, and by connecting modules through communication buffers. Although it is not obvious to make a strict comparison with DC-systems, one may observe that the restrictions imposed on the buffers ('private destination' principle, preservation of the equal conflict sets of modules by buffers) in (DS)*SP, does not allow to describe systems as the DC-system of Figure 4, since more relaxed constraints of conflict/synchronisation are permitted.

The strong-connectivity of N* (Cf. definition 4.5 of Root net systems) is sufficient but not a necessary condition to ensure the DC-property (and the membership to DC-systems). By adding appropriate structure to the subnet induced by root places (considered as modules), one can provide methods for the synthesis of live DC-systems. From this point of view, we can revisit the building process of the class of modular systems (DS)*SP.

5 Computational Methods

In this section, we study the hard problem of membership to DC-systems from a computational point of view. Thus, we present death propagation schemes according to the death causality relation previously introduced and based on 'topological' features of nets. Then, we present an algorithm that allows determining the causality set associated with a given transition.

Let us consider a P/T system which is deadlock-free but not live. We state the following assertion: if a transition t is assumed to be dead for M*, then all the output transitions of its root places are also dead for M*. In other words, if the root place of a dead transition t is also an input place of another transition t', then transition t' is necessarily dead. Hence, the root place plays the role of a *death (non-liveness) propagation* means.

Proposition 5.1 *death propagation by root*

Let <N, M_0> be a P/T system deadlock-free but not live, and $M^* \in Acc(M_0)$ be a singular marking (by proposition 3.1). Let T_D be the subset of dead transitions for M* :

$$t \in T_D \text{ and t admits a root place r} \Rightarrow r^\bullet \subseteq T_D \qquad \text{(i.e. } r^\bullet \cap T_L = \varnothing\text{).}$$

The previous proposition is relevant only when a root place of transition t has more than one output transition (i.e. has other output transitions different from t). Indeed, if the root place has only t as an output transition, the result is obvious.

Example

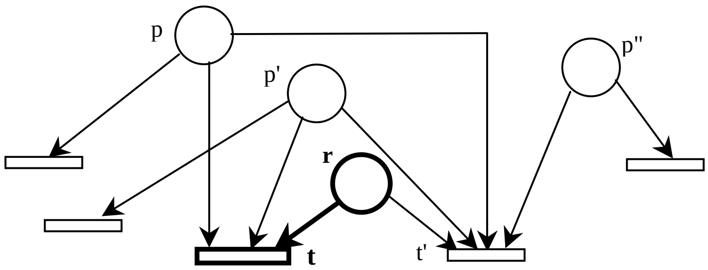

Figure 5 : subnet with a root place shared by two transitions (r is a root place of t ; if t is dead then t' is dead)

The proof of this proposition relies on the definition of root and on the following fact : transition t is dead for M* signifies that its input places remain always *not sufficiently marked* to enable t, for all reachable markings from M*. Moreover, if t admits a root place r which is also an input place of a transition t' (see example of Figure 5), then all input places of t are also input places of t'. This leads us to deduce that input places of t' also remain *not sufficiently marked* to enable t', for all reachable markings from M*. So, transition t' is also dead for M*.

Proof:

First, let's prove that : $^\bullet t \subseteq {}^\bullet t'$ for every t' of $r^\bullet$.

As $\forall p \in {}^\bullet t$, $r^\bullet \subseteq p^\bullet$ (by definition of root), then $\forall t' \in r^\bullet$, $t' \in p^\bullet$ ($\forall p \in {}^\bullet t$) ;

then $\forall p \in {}^\bullet t$, $p \in {}^\bullet t'$ which means $^\bullet t \subseteq {}^\bullet t'$ for every t' of $r^\bullet$.

Now, let's recall the following :

$$\text{t is dead for M*} \quad \Leftrightarrow \quad \forall M \in Acc(M^*), \exists p \in {}^\bullet t : M(p) < V(p,t)$$

As $^\bullet t \subseteq {}^\bullet t'$ (for every t' of $r^\bullet$), then $\forall M \in Acc(M^*), \exists p \in {}^\bullet t' : M(p) < V(p,t')$

$\Rightarrow$ t' is dead for M*.

We can also state the following : if all input transitions of a place are dead for a given singular marking, then all its output transitions are dead. Intuitively, if all transitions *producing* tokens in a given place are dead, this place will be empty, and then not sufficiently marked to enable its output transitions that act as *consuming* transitions over the concerned place. Let's call this statement the *death (non-liveness) propagation by pipe*. Figure 6 illustrates such a configuration.

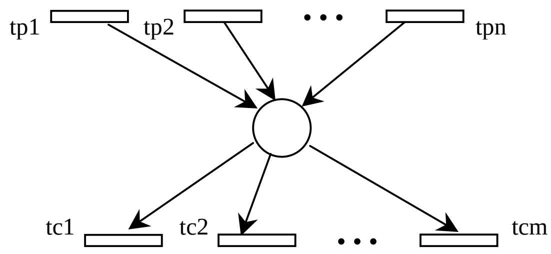

Figure 6 : subnet illustrating a 'pipe' configuration
(if tp_1, …, tp_n are dead, then tc_1,…, tc_m are dead)

Proposition 5.2 *death propagation by pipe*

Let <N, M_0> be a P/T system deadlock-free but not live, and $M^* \in Acc(M_0)$ be a singular marking. Let T_D be the subset of dead transitions for M* :

$$^\bullet p \subseteq T_D \quad \Rightarrow \quad p^\bullet \subseteq T_D \qquad (\forall p \in P).$$

Proof:

Suppose $^\bullet p \subseteq T_D$ and $p^\bullet \cap T_L \neq \emptyset$. Then, $\exists t \in p^\bullet$ such that t is live.

Since $^\bullet p \subseteq T_D$, one can necessarily reach a marking $M^{**} \in Acc(M^*)$ for which t becomes dead. This contradicts that t is live and the maximality of T_D.

Finally, we state the following : given a *bounded* place p, if all output transitions of this place p are dead, then all its input transitions are necessarily dead, otherwise place p cannot be bounded. Let's call this statement the *death (non-liveness) propagation by bounded place.* Figure 7 illustrates such a configuration.

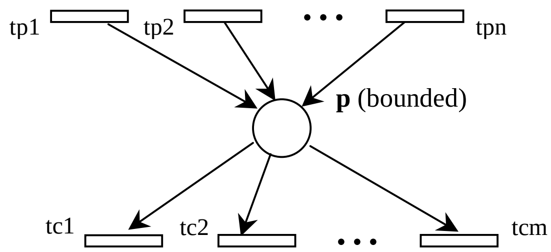

Figure 7 : subnet illustrating a death propagation by bounded place
(if tc_1, …, tc_m are dead, then tp_1,…, tp_n are dead)

Proposition 5.3 *death propagation by bounded place*

Let <N, M_0> be a P/T system deadlock-free but not live, and p be a *bounded* place of P. Let $M^* \in Acc(M_0)$ be a singular marking and T_D be the subset of dead transitions for M* :

$$p^\bullet \subseteq T_D \quad \Rightarrow \quad {}^\bullet p \subseteq T_D$$

Proof: Suppose $p^\bullet \subseteq T_D$ and $\exists t \in {}^\bullet p : t \in T_L$.

As $p^\bullet \subseteq T_D$, M(p) cannot decrease for all markings $M \in Acc(M^*)$.

As $t \in T_L$, M(p) will continuously increase for markings $M \in Acc(M^*)$.

This contradicts the boundedness hypothesis of p.

Let's show how the death causality set D(t), associated with a given transition t, can be *partially* determined using rules deduced from the topological based properties of death propagation provided by propositions 5.1, 5.2 and 5.3. As these rules do not ensure that all elements of D(t) are determined, the algorithm can only be used as a sufficient test for a system to be a DC-system.

Initially, we set D(t) = {t} , as t is assumed to be dead. Then, we apply the following three rules :

R1. Let p be a root place of t: $t \in D(t) \Rightarrow p^\bullet \subseteq D(t)$ (death propagation by root)

R2. Let p be a place of P: ${}^\bullet p \subseteq D(t) \Rightarrow p^\bullet \subseteq D(t)$ (death propagation by pipe)

R3. Let p be a bounded place: $p^\bullet \subseteq D(t) \Rightarrow {}^\bullet p \subseteq D(t)$ (death propagation by bounded place)

One can easily check the correctness of these rules through propositions 5.1, 5.2 and 5.3. We present now an algorithm (in Pascal style) for the computation of D(t). Its complexity is similar to classical graph traversal algorithms (O $(|P| \times |T|)$).

Algorithm *Computing_Dt*

input: a transition t; // t is assumed to be dead

output: Dt a set of transitions; // D(t)

Variable: Dt_marked a set of transitions; // transitions for which R1, R2 or R3 have been applied

```
begin
    Dt :={t} ; Dt_marked := ∅ ;
    while ( Dt \ Dt_marked ≠ ∅ ) do
        get t from (Dt \ Dt_marked) ;
        Dt_marked := Dt_marked ∪ {t}; // make t marked
        If r is a root place of t then Dt := Dt ∪ r• ;        // application of R1
        for each (p ∈ t•) do
            if (•p ⊆ Dt) then Dt := Dt ∪ p• ;                // application of R2
        od
```

```
        for each (p ∈ •t) such that (p is bounded) do
            if (p• ⊆ Dt) then Dt := Dt ∪ •p ;          // application of R3
        od
    od
end
```

One may apply this algorithm on P/T system of Figure 4. Then, we can state that the only non-ordered transition is t3. Computing D(t3) leads us to deduce that t4 is dead (by R2 on d), t2 is dead (by R2 on c), and t1 is dead (by R2 on f). As D(t3)=T (and T_{no} ={t3}), the P/T system is a DC-system.

6 Conclusion

In this paper we introduced a non-liveness causality relation based on structure theory of Place/transition systems. In particular we present a refined characterisation of the non-liveness condition under controlled siphon property. This topological result leads us to revisit from a new perspective well known results, and make applicable the structural analysis techniques for a large class of P/T systems, called DC-systems, where the interplay between conflict and synchronisation is relaxed but for which equivalence between deadlock-freeness and liveness holds.

Acknowledgements

I would thank Professor K. Barkaoui who suggested this issue and with whom the present work has been initiated.

References

[Aalst et al., 98] Wil van der Aalst, Ekkart Kindler, and Jorg Desel. "*Beyond asymmetric choice: A note on some extensions"*. Petri Net Newsletter, 55:3-13, October 1998.

[Barkaoui and Zouari 03] Barkaoui, K., Zouari, B. : "*On the Equivalence between Liveness and Deadlock Freeness in Petri Nets".* 6th International Symposium on Programming and Systems (ISPS'03), Alger, may 2003.

[Barkaoui et al., 95] K. Barkaoui, J.F. Peyre, "*On Liveness and Controlled Siphon Property in Nets"*, ICATPN, LNCS n° 1091, Springer 1996.

[Commoner et al., 71] F. Commoner, A.W. Holt, S. Even, and A. Pnueli. "*Marked directed graphs"*. Journal of Computer Systems Science, 5:72-79, 1971.

[Desel, 92] J. Desel, "*A proof of the rank theorem for extended free choice nets"*, ICATPN, LNCS n°616, 1992.

[Desel et al., 95] J. Dezel and J. Esparza. "*Free Choice Petri Nets"*. Cambridge Tracts in Theoretical Computer Science. Vol 40, 1995.

[Esparza et al., 92] J. Esparza, M. Silva, "*A Polynomial time algorithm to decide liveness of bounded free-choice nets"*, TCS, n°102,1992.

[Lautenbach et al; 94] K. Lautenbach, H. Ridder, "*Liveness in bounded Petri nets which are covered by T-invariants*",. ICATPN, LNCS n° 815, Springer 1994.

[Reisig, 91] W.Reisig , "*Petri Nets*". Springer-Verlag, 2nd edition, 1991.

[Recalde et al., 01] L. Recalde, E. Teruel, M. Silva, "*Structure Theory of multi-level deterministically synchronised processes*", TCS, Vol (254)1 ; pp 1-33, 2001.

[Souissi, 93] Y. Souissi, "*Deterministic systems of sequential processes: a class of structured Petri nets*", ICATPN, LNCS n° 674, Springer Verlag, 1993.

[Teruel et al., 96] E. Teruel, M. Silva, "*Structure Theory of equal conflict systems*", TCS, vol (153)1 ; pp 271-300, 1996.

[Teruel et al., 97] E. Teruel, J.M. Colom, and M. Silva. "*Choice-free Petri nets : A model for deterministic concurrent systems with bulk services and arrivals*". IEEE Trans. On Systems Man and Cybernetics, 27(1): 73-83, 1997.