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A Constructive Approach to Sylvester's Conjecture[1]

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Abstract: Sylvester's conjecture states that, given n distinct noncollinear points in a plane, there exists a connecting line of two of the points such that no other point is incident with the line. First a proof is given of the six-point Sylvester conjecture from a constructive axiomatization of plane incidence geometry. Next ordering principles are studied that are needed for the seven-point case. This results in a symmetrically ordered plane affine geometry. A corollary is the axiom of complete quadrangles. Finally, it is shown that the problem admits of an arithmetic translation by which Sylvester's conjcture is decidable for any n.

Key Words: Sylvester's conjecture, constructive geometry, ordered geometry
Category: F.4.1

1 Introduction

Sylvester's conjecture states that, given n distinct noncollinear points in a plane, there exists a connecting line of two of the points such that no other point is incident with the line. It is known that for seven points, the conjecture does not follow from usual axioms of plane incidence geometry. We first give a proof up to six points, from a constructive axiomatization of incidence geometry. An ordered affine geometry is presented and the seven-point case derived essentially from the principle: A point apart from a given line cannot be on both sides of it. The axiom of complete quadrangles is a corollary to our result.

In our constructive approach, we use constructions instead of existence axioms. Thus, the connecting line is introduced through a function that takes two points *and a proof that they are distinct* as arguments, and gives the line as value. Because of the condition, properties of constructions cannot be expressed by ordinary first order logic. But in Sylvester's conjecture, all points are pairwise distinct by assumption, and for this reason we can find a translation of the conjecture that only uses propositional logic. It follows that for any given number of points, constructive provability of Sylvester's conjecture can be algorithmically decided. In particular, independence of the seven-point conjecture from the usual axioms, without ordering principles, can be established by proof-theoretical methods.

The motivation for a constructive development is that constructivity supports computability: Functions effecting geometric constructions are algorithmic, and

[1] C. S. Calude, H. Ishihara (eds.). *Constructivity, Computability, and Logic. A Collection of Papers in Honour of the 60th Birthday of Douglas Bridges.*

so are proofs, in particular, the existence proof for Sylvester's conjecture gives an algorithm for finding a two-point line.

2 Constructive incidence geometry

The standard axiomatic approach to plane incidence geometry is: There is a set of points, a set of lines, equality relations for points and for lines, and an incidence relation between points and lines. Equal objects can be substituted in the incidence relation. Next, to any two unequal points, there exists a line with which the two points are incident, and to any two unequal lines, there exists a point incident with both. Uniqueness of these *connecting lines* and *intersection points* is best taken care of by *Skolem's axiom*: If two points are incident with two lines, the points or lines are equal.

Equality and incidence are ideal relations. We can start instead from the relations of *distinct* points, *distinct* lines, and *apartness* of a point from a line. These relations have a finitary computational meaning: Say, if a point is apart from a line in the real plane, a finite approximation will tell this, but if it is incident, the point has to be computed to infinite precision to verify the incidence. Thus, constructively the classical equality and incidence relations should come out as negations: With $a, b, c, \ldots$ denoting points, and $l, m, n, \ldots$ lines, we define

$$\begin{aligned} a = b \quad &=_{df} \ \sim a \neq b, \\ l = m \quad &=_{df} \ \sim l \neq m, \\ I(a,l) \quad &=_{df} \ \sim A(a,l). \end{aligned}$$

where the primitive relations $a \neq b$, $l \neq m$, and $A(a,l)$ stand for, a is distinct from b, l is distinct from m, and a is apart from l.

We use the abbreviations

$$A(a \cdot b, l) = A(a,l) \& A(b,l), \qquad A(a|b, l) = A(a,l) \vee A(b,l),$$

and similarly for the second argument. For incidence, the abbreviations are to be unfolded as for apartness, so $I(a \cdot b, l)$ for example, means $I(a,l) \& I(b,l)$.

Instead of existence axioms, we use construction postulates:

If $a \neq b$, $ln(a,b)$ is a line,
if $l \neq m$, $pt(l,m)$ is a point.

Axioms of constructive incidence geometry:

Ia. $\sim a \neq a$, $\sim l \neq l$.
Ib. $a \neq b \rightarrow a \neq c \vee b \neq c$, $l \neq m \rightarrow l \neq n \vee m \neq n$.

II. $a \neq b \rightarrow \sim A(a, ln(a,b))$, $a \neq b \rightarrow \sim A(b, ln(a,b))$,
$l \neq m \rightarrow \sim A(pt(l,m), l)$, $l \neq m \rightarrow \sim A(pt(l,m), m)$.

III. $a \neq b \;\&\; l \neq m \rightarrow A(a,l) \vee A(a,m) \vee A(b,l) \vee A(b,m)$.

IV. $A(a,l) \rightarrow a \neq b \vee A(b,l)$,
$A(a,l) \rightarrow l \neq m \vee A(a,m)$.

Contraposition of Ib is transitivity of equality. Substitution of a for c (l for m) in Ib gives $a \neq b \rightarrow b \neq a$ (and $l \neq m \rightarrow m \neq l$).

Axioms in group II express the ideal properties of constructed objects. III is a general principle equivalent to the uniqueness of our two constructions.

Taking contrapositions, IV gives the classical principles of substitution of equals.

There are no existence axioms in I–IV, but we shall usually assume given geometric objects to the same effect, say, at least three given noncollinear points. A detailed exposition of constructive geometry is given in [von Plato 1995]. Through constructive axiomatization, proofs of theorems as well as solutions to problems are algorithms that take the given as data and compute the thing sought as value. Constructivity here has the consequence that the geometrical algorithms are provably terminating.

3 Constructive proof of Sylvester's conjecture up to six points

We shall express the assumption of noncollinearity as follows:

Sylvester's hypothesis for n distinct points $a_1, ..., a_n$: For any line x, $A(a_1|...|a_n, x)$. Abbreviate this as $SH(n)$. Collinearity can be defined as the negation of noncollinearity, and the latter notion is indeed the positive one, despite the terminology. We could equally well call *planar* points noncollinear in our sense.

Sylvester's claim for n points is that there are at least two distinct points a_i, a_j such that $A(a_k, ln(a_i, a_j))$ whenever $a_k \neq a_i, a_j$. Abbreviate this as $SC(n)$.

Lemmas needed in the proof: The following results, easy consequences from the axioms and derived in [von Plato 1995], are constantly used in the proof.

Lemma 1. $A(c, ln(a,b)) \rightarrow a \neq c \;\&\; b \neq c$.

Lemma 2. $A(c, ln(a,b)) \rightarrow A(a, ln(b,c))$.

Lemma 3. $A(c, ln(a,b)) \rightarrow ln(a,b) \neq ln(b,c)$.

Because we deal with less than ten points, it will be helpful to use the following notation:

$$A(k, ij) =_{df} A(a_k, ln(a_i, a_j)).$$

Further, since $ln(a, b) = ln(b, a)$ [see von Plato 1995, theorem 5.1], we can make the

Convention: *Write* $A(k, ij)$ *always with* $i < j$.

Now each proposition of form $A(k, ij)$ gives by lemma 2 two others of the same form. The method of the following proof is always to derive the distinctness of points by lemmas 1 and 2, the distinctness of lines by lemma 3, and then to apply axiom III which gives us a number of cases. It remains to check that Sylvester's claim is verified in each case.

Theorem 4. $SH(n) \rightarrow SC(n)$ *if* $n \leq 6$.

Proof. By our convention $i < j$ if $A(k, ij)$, so we have $1 \leq i < j \leq n$. Therefore $n \geq 2$. For $n = 2$, $SC(2)$ is vacuously satisfied. For $n = 3$, $SH(3)$ gives $A(1|2|3, 12)$, so by axiom II, $A(3, 12)$. We shall apply axiom II without explicit mention below.

$n = 4$: $A(3|4, 12)$ by $SH(4)$. Case 1: $A(3, 12)$. Then $A(2, 13)$ and $A(1, 23)$ by lemma 3, and $12 \neq 13$ by lemma 3. Therefore $A(4, 12|13)$ by axiom III. Case 1.1: $A(4, 12)$. Proof is finished. Case 1.2: $A(4, 13)$. Proof is finished. Case 2. $A(4, 12)$. Then $A(2, 14)$ and $A(1, 24)$ by lemma 2 and $A(3, 14|24)$ by axiom III. Case 2.1: $A(3, 14)$. Proof is finished. Case 2.2: $A(3, 24)$. Proof is finished.

$n = 5$: We shall just list the cases to make the presentation shorter. By $SH(5)$, $A(3|4|5, 12)$.

Case 1: $A(3, 12)$. Axiom III gives $A(4, 12|13)$. Case 1.1: $A(4, 12)$. Axiom III gives $A(5, 12|13)$. Case 1.1.1: $A(5, 12)$. Case 1.1.2: $A(5, 13)$. Axiom III gives $A(4, 13|23)$. Case 1.1.2.1: $A(4, 13)$. Case 1.1.2.2: $A(4, 23)$. Axiom III gives $A(5, 12|23)$. Case 1.1.2.2.1: $A(5, 12)$. Case 1.1.2.2.2: $A(5, 23)$. Case 1.2. $A(4, 13)$. Axiom III gives $A(5, 12|13)$. Case 1.2.1: $A(5,12)$. Axiom III gives $A(4,12|23)$. Case 1.2.1.1: $A(4,12)$. Case 1.2.1.2: $A(4, 23)$. Axiom III gives $A(5, 13|23)$. Case 1.2.1.2.1: $A(5, 13)$. Case 1.2.1.2.2: $A(5, 23)$. Case 1.2.2: $A(5, 13)$.

Case 2: $A(4, 12)$ is like previous with 3 and 4 interchanged.

Case 3: $A(5, 12)$ the same.

Instead of continuing the proof for $n = 6$ in the same way, we shall just describe how it can be constructed, for the actual proof is far too long to be included here. This will also illustrate further the mechanism of case generation in the proof. First list all lines in the order

12
13, 23
14, 24, 34
15, 25, 35, 45

16, 26, 36, 46, 56.

Start from the assumption $A(3|4|5|6, 12)$. First case is for the point with the smallest number, here point 3. Take the next point 4, find the first line ij with $i, j \neq 4$, find the first line ik or kj where $k \neq 4$. Say it is ik. Then by axiom III, $A(4, ij|ik)$, so we have two cases. Repeat the above in both. Following this algorithm, and keeping track of what lemma 2 gives, one always finds a line with just two of the points incident with it.

It is of some interest to compare the above constructive proof to a classical one. With four points, point 4 either is apart from all the three lines 12, 13, and 23, in which case there is nothing more to prove, or it is incident with at least one of them. Since the points are assumed distinct, it follows that 4 is incident with just one of the lines, and therefore we can conclude that it is apart from the remaining two. In the constructive proof, we have to get along with the weaker conclusion that a point is apart from one of any two lines if it is distinct from their intersection point. This generates the great number of cases in the constructive proof.

Lemma 2 that was constantly used in the proof, is known as the 'triangle axiom'. As shown in [von Plato 1995], corollary 4.10, it has a dual formulation in terms of intersection points: $A(pt(l,m), n) \rightarrow A(pt(m,n), l)$. This will at once give a dual to Sylvester's conjecture, namely that for any n pairwise distinct nonconcurrent lines, there is at least one intersection point of two lines not incident with any of the other lines.

Dual SH(n) for n distinct lines: For any point x, $A(x, l_1|\dots|l_n)$.

The *dual* $SC(n)$ now is that there are at least two distinct lines $l_i,\ l_j$ such that $A(pt(l_i, l_j), l_k)$ whenever $l_k \neq l_i, l_j$. We have the

Corollary 5. $DSH(n) \rightarrow DSC(n)$ *for* $n \leq 6$.

4 The seven-point conjecture; refinement of the apartness relation

Continuing for seven points the algorithm described in the above proof, one would find that one runs out of lines without finding in all cases a line with just two points on it. (This, though, may not be the most effective way of finding it out, as nested subcases are produced up to 18 times.) All such paths are variants of the following one: There are proofs of

$A(3,12),\ A(4,13),\ A(4,23),\ A(5,12),\ A(5,13),\ A(6,12),\ A(6,23),$
$A(7,12),\ A(7,13),\ A(7,23),\ A(7,14),\ A(7,24),\ A(7,35),$
$A(7,45),\ A(7,36),\ A(7,46),\ A(7,25),\ A(7,16),\ A(7,56).$

One cannot prove $A(4,56)$, and indeed, the seven-point Sylvester conjecture is known to be independent of the usual axioms of incidence geometry (as pointed out to us in correspondence by the late Prof. E.W. Dijkstra). The following classical "worst case" argument for seven points will show the difficulty: As in the previous section's classical proof sketch, assume $A(3,12)$. In the worst case, add points 4,5,6, with

$I(4,12),\ I(5,13),\ I(6,23),$

so the original three lines have more than two points. But the new lines

34, 25, 16, 45, 46, 56

were generated, and at least one of them has just two points incident with it by theorem 4. From $I(4,12)$, $I(5,13)$ and $I(6,23)$ it is easy to prove that

$A(5\cdot 6,34)$, $A(4\cdot 6,25)$, and $A(4\cdot 5,16)$.

Since $12 = 14 = 24$, $13 = 15 = 35$ and $23 = 26 = 36$ (from axiom III, or see [von Plato 1995], lemma 4.5), we have by substitution and lemma 2 also

$A(1\cdot 2,34)$, $A(1\cdot 3,25)$, and $A(2\cdot 3,16)$.

Adding point 7, the worst case is $I(7,34\cdot 25\cdot 16)$ or else we have a line with just two points on it. It is helpful to draw a diagram with the three lines as, say, the altitudes of the triangle 123. There is a well known figure (model) of seven-point projective geometry, with a circular "line" inscribed in a triangle and containing the points 4, 5 and 6, and thereby falsifying Sylvester's conjecture. One sees that with straight lines, in order to prove $I(4,56)$ for such a figure, point 4 would have to be on a different side of line 23. This idea shall now be implemented.

We shall refine the apartness relation $A(a,l)$ into $L(a,l) \vee R(a,l)$. (For *L*eft, *R*ight.) As with all such refinements, one starts from the new basic relations and derives the properties of the old basic relation. Secondly, the new concepts in the refinement must be chosen incompatible (see [von Plato 1996] for more details on how refinement affects axiomatization).

We used the condition $l \neq m$ in the construction postulate for intersection points. We can replace this condition by the *convergence* of two lines, where convergence is a relation the negation of which is the parallelism of two lines. Quite analogously to the axioms for distinct points or distinct lines, we make the

Definition 6. $l \parallel m =_{df} \sim l \nparallel m$.

Next we define apartness:

Definition 7. $A(a,l) =_{df} L(a,l) \vee R(a,l)$.

Now the axioms proper: To the general axioms for basic concepts in group I we must add the two axioms for line convergence, and the incompatibility of L and R. Axioms in groups II-III that contain the apartness relation, are merely rewritten with $L(a,l) \vee R(a,l)$ in place of $A(a,l)$, except that we use the condition of convergence in the axioms expressing the ideal properties of intersection points. To the substitution axioms we add one for line convergence. The substitution axiom for points in the apartness relation, IVa, is stronger. The corresponding principle for lines, IVb, instead, cannot be strengthened with the present concepts in the way of IVa. We shall instead add principles for the ordering properties of constructed lines.

OIa. $\sim a \neq a,\ \sim l \neq l,\ \sim l \nparallel l.$

OIb. $a \neq b \rightarrow a \neq c \vee b \neq c,$
$l \neq m \rightarrow l \neq n \vee m \neq n,$
$l \nparallel m \rightarrow l \nparallel n \vee m \nparallel n.$

OIc. $\sim(L(a,l)\ \&\ R(a,l)).$

OII. $a \neq b \rightarrow \sim(L(a, ln(a,b)) \vee R(a, ln(a,b))),$
$a \neq b \rightarrow \sim(L(b, ln(a,b)) \vee R(b, ln(a,b))),$
$l \nparallel m \rightarrow \sim(L(pt(l,m),l) \vee R(pt(l,m),l)),$
$l \nparallel m \rightarrow \sim(L(pt(l,m),m) \vee R(pt(l,m),m)).$

OIII. $a \neq b\ \&\ l \neq m \rightarrow L(a|b, l|m) \vee R(a|b, l|m).$

OIVa. $L(a,l) \rightarrow a \neq b \vee L(b,l),$
$R(a,l) \rightarrow a \neq b \vee R(b,l),$

OIVb. $L(a,l) \rightarrow l \neq m \vee L(a,m) \vee R(a,m),$
$R(a,l) \rightarrow l \neq m \vee L(a,m) \vee R(a,m).$

OIVc. $l \nparallel m \rightarrow m \neq n \vee l \nparallel n.$

OV. $a \neq b\ \&\ L(c, ln(a,b)) \rightarrow R(b, ln(a,c)),$
$a \neq c\ \&\ R(b, ln(a,c)) \rightarrow L(a, ln(b,c)).$

It follows that parallelism of two lines is an equivalence relation. By the requirement of convergence instead of only distinctness, the intersection point construction can be applied in affine and other geometries where lines behave in a more straight way than in the projective case. Uniqueness of the intersection point follows naturally: First, as we have one more basic concept, substitution of equals in it must be assumed. This was effected by adding axiom OIVc. Its contraposition is the classical substitution principle of equal lines in the parallelism relation. It follows at once that $l \nparallel m \rightarrow l \neq m$, and now the crucial constructive Skolem axiom can be applied to prove uniqueness. The converse,

$$l \neq m \rightarrow l \nparallel m$$

is the constructive version of the characteristic axiom of projective geometry, *viz.*, all parallels are to be equated.

Axioms OV for constructed lines can be seen as ordered versions of the usual triangle axioms. In contrast to all the other axioms, their postulation is not based on any evident principles of constructive axiomatization (but see [von Plato 1998] for a remedy).

The axioms have also interesting general consequences. Just to give an example, assume $L(a, l)$, $R(b, l)$. It follows at once from OIc and OIVa that $a \neq b$.

Definition 8. $Div(l, a, b) =_{df} (L(a, l) \ \& \ R(b, l)) \vee (R(a, l) \ \& \ L(b, l))$.

$Div(l, a, b)$ expresses that points a, b are on different sides of line l, or *divided* by l.

Theorem 9. $Div(l, a, b) \ \& \ A(c, l) \rightarrow Div(l, a, c) \vee Div(l, b, c)$.

Proof. By assumption, $(L(a, l) \ \& \ R(b, l)) \vee (R(a, l) \ \& \ L(b, l))$ and $L(c, l) \vee R(c, l)$. In each of the four cases, $Div(l, a, c)$ or $Div(l, b, c)$ follows.

Theorem 9 is usually called the axiom of Pasch. The observation that it is provable by propositional logic from our axioms and definition of Div is due to Per Martin-Löf. The result can be strengthened as follows:

Theorem 10. $Div(l, a, b) \ \& \ Div(l, a, c) \rightarrow \sim Div(l, b, c)$.

Proof. Assume $Div(l, b, c)$. By definition, we have eight cases of $L-R$ relations, starting with

$$L(a, l) \ \& \ R(b, l) \ \& \ L(a, l) \ \& \ R(c, l) \ \& \ L(b, l) \ \& \ R(c, l).$$

Each of these cases contains an $L-R$ pair that is incompatible by axiom OIc. Therefore $\sim Div(l, b, c)$.

Theorem 10 shows that no line can divide all the three pairs of points of a triangle.

Our counterexample to the "worst case" of the seven-point Sylvester's conjecture required a line of precisely the kind denied by theorem 10. From this theorem we get that

$$Div(56, 1, 3) \ \& \ Div(56, 2, 3) \rightarrow \sim Div(56, 1, 2).$$

Then, points 1 and 2 are on the same side of line 56. Point 4 is between points 1 and 2, so we have the

Proposition 11. *The seven point Sylvester conjecture follows from the convexity of the half-plane.*

As to convexity, there are two possibilities: taking betweenness as primitive and defining convexity in the usual way, or defining betweenness and convexity in a suitable way and proving the old definitional relation as a theorem. The latter approach is the more useful one here. Betweenness is defined as: Assume $I(c, ln(a, b))$, and let d be any point such that $A(d, ln(a, b))$. Then

$$Bet(a, c, b) =_{df} Div(ln(c, d), a, b).$$

$Div(l, a, b)$ is symmetric in a and b, therefore $Bet(a, c, b) \rightarrow Bet(b, c, a)$. Other properties such as $Bet(a, c, b) \rightarrow \sim Bet(c, a, b)$ are also straightforward. It is clear that we have now gone past projective geometry that does not admit of a proper betweenness relation.

5 The parallel construction and its properties

We want to establish a connection between order on a line and in the plane, by showing that the half-plane is convex. As shown by the seven-point model of projective geometry with the circular line, such convexity is an affine property.

The basic concept is the convergence of two lines, with parallelism defined as its negation. For the latter, there is a construction postulate that assumes given a line l and a point a:

$par(l, a)$ is a line.

The properties are

AF1. $\sim(L(a, par(l, a)) \vee R(a, par(l, a)))$.

AF2. $\sim par(l, a) \nparallel l$.

AF3. $a \neq b \;\&\; L(a, l) \rightarrow L(b, l) \vee l \nparallel ln(a, b)$,
$a \neq b \;\&\; R(a, l) \rightarrow R(b, l) \vee l \nparallel ln(a, b)$.

AF4. $L(a, l) \;\&\; L(b, par(l, a)) \rightarrow L(b, l)$,
$R(a, l) \;\&\; R(b, par(l, a)) \rightarrow R(b, l)$.

AF5. $L(b, par(l, a)) \leftrightarrow R(a, par(l, b))$.

The first two principles express the ideal properties of constructed parallels. The third one is the uniqueness axiom for the parallel line construction: Assume $a \neq b$. If $L(a, l)$, the incidence $I(b, par(l, a))$ implies that $ln(a, b) = par(l, a)$. It also follows that $L(b, l)$. This consequence of axiom AF3 is a very traditional rendering of the property of parallelism: No point incident with $par(l, a)$ can be incident with l, provided that $A(a, l)$. More interestingly, if $L(a, l)$ and $R(b, l)$, it follows that $a \neq b$ as mentioned previously, and axiom AF3 now gives the Jordan property of the half-plane: If two points are on different sides of a line, their connecting line intersects the dividing line. Axiom AF4 permits the substitution of lines in the parallel construction. As with connecting lines, we must postulate a principle that gives the connection between the parallel line construction and the ordering of the plane. This was done in AF5.

6 Convexity of the half-plane

We are now in a position to establish the connection between order on a line and the ordering of the plane through refinement of the apartness relation.

Theorem 12. $L(a \cdot b, l)$ & $Bet(a, c, b) \rightarrow L(c, l)$.

Proof. By AF3, $L(c, l) \vee l \not\parallel ln(a, c)$. In the first case, proof is finished. Assume therefore $l \not\parallel ln(a, c)$. Then $par(l, c) \not\parallel ln(a, c)$ by substitution of parallels in line convergence. Now $A(a \cdot b, par(l, c))$ follows. By definition of $Bet(a, c, b)$, we have $Div(par(l, c), a, b)$, so there are two cases: Case 1: $R(a, par(l, c))$ and $L(b, par(l, c))$. By axiom AF4, $R(a, par(l, c))$ gives $L(c, par(l, a))$. Therefore $L(c, l)$ by AF5. Case 2: $L(a, par(l, c))$ and $R(b, par(l, c))$. Proof analogous to case 1.

Convexity of a set of points S is defined as follows: Let l be any line such that if $a \in S$, $A(a, l)$. Let $a, b \in S$. Now set

$$Convex(S) =_{df} L(a \cdot b, l) \vee R(a \cdot b, l).$$

The set of points of an open segment is defined as

$$Seg(a, b) =_{df} \{x | Bet(a, x, b)\}.$$

Convexity of a line segment follows at once from theorem 12:

Corollary 13. $Convex(Seg(a, b))$.

Corollary 14. *Let* $1, ..., 6$ *be distinct points with* $L(3, 12)$. *Then*

$$Bet(1, 4, 2) \ \& \ Bet(1, 5, 3) \ \& \ Bet(2, 6, 3) \rightarrow A(4, 56).$$

Proof. By assumption, $I(5, 13)$. Then $A(6, 13)$ follows easily, and the assumption $Bet(1, 5, 3)$ now gives $Div(56, 1, 3)$. Similarly $Div(56, 2, 3)$. Since $A(1 \cdot 2 \cdot 3, 56)$, in particular $L(3, 56) \vee R(3, 56)$. If $L(3, 56)$, then $R(1 \cdot 2, 56)$, and if $R(3, 56)$, then $L(1 \cdot 2, 56)$. In the first case, $Bet(4, 1, 2)$ gives by theorem 12 $R(4, 56)$, in the second it gives $L(4, 56)$.

By proposition 11, we have proved the seven-point Sylvester conjecture in the "worst case". A complete constructive reduction of the general case has too many details to be presented here.

7 The axiom of complete quadrangles

We shall now derive from the seven-point Sylvester conjecture the *axiom of complete quadrangles*. First define a complete quadrangle as follows:

Definition 15. $Quad(a,b,c,d) =_{df} a \neq b \;\&\; c \neq d \;\&\; A(c \cdot d, ln(a,b)) \;\&\; A(a \cdot b, ln(c,d))$.

It follows that each of the four sides has just two of the points a, b, c, d incident with it. The *diagonal points* of the quadrangle are defined as

$p_1 =_{df} pt(ln(a,b), ln(c,d))$,
$p_2 =_{df} pt(ln(b,c), ln(d,a))$,
$p_3 =_{df} pt(ln(a,c), ln(b,d))$.

The axiom of complete quadrangles is the claim that $p_1p_2p_3$ is a proper triangle (the 'diagonal triangle'). Note that by lemma 3, the condition $A(c, ln(a,b))$ implies $ln(a,b) \neq ln(c,d)$, and so on, so this condition is sufficient for the intersection point construction to be applicable. In the affine situation, we have to assume $ln(a,b) \nparallel ln(c,d)$ and $ln(b,c) \nparallel ln(d,a)$. (The third condition can be derived.)

Corollary 16. $Quad(a,b,c,d) \rightarrow A(p_3, ln(p_1,p_2))$.

Proof. The seven points $a, b, c, d, p_1, p_2, p_3$ are noncollinear, and all the points are distinct. Each one of the sides $ln(a,b)$, $ln(b,c)$, $ln(c,d)$ and $ln(d,a)$ has at least three of the points incident with it by construction. Same with the diagonals $ln(a,c)$ and $ln(b,d)$. By the seven-point Sylvester conjecture, at least one of the lines $ln(p_1,p_2)$, $ln(p_2,p_3)$ and $ln(p_1,p_3)$ has just two of the seven points incident with it. In each case, $A(p_3, ln(p_1,p_2))$.

Quadrangles can be of two kinds: In terms of L and R, we can define a quadrangle to be convex if none of its four sides divides the remaining two points, and concave in the contrary case. A concave quadrangle gives precisely the situation of the seven-point Sylvester conjecture, and the axiom follows. For a convex quadrangle, the axiom can also be derived from a more general result according to which one of its diagonal points is inside the quadrangle, with the connecting line of the other two strictly supporting it. To get this result, one defines the set of points of a quadrangle in the obvious way, and shows that this combinatorial definition of convexity implies convexity in the sense of the definition in the previous section, analogously to the case of line segments in corollary 13.

8 Arithmetic translation of Sylvester's conjecture

The practice of using numbers as identifiers for points can be turned into an arithmetic translation of Sylvester's conjecture. Equality of points is not decidable in general [cf. von Plato 1995, p. 170], but an arithmetic translation is still possible because we assume all the points distinct and thereby avoid that decidability problem. There will be two primitive relations obeying suitable axioms, with

Variabes $x, y, z, ...$ ranging over $1, ..., n$.

Three-place relations $LW,\ RW$ over $1, \ldots, n$.

Definition 17. $L(x, y, z) =_{df} \sim y = z \ \&\ LW(x, y, z)$,
$R(x, y, z) =_{df} \sim y = z \ \&\ RW(x, y, z)$.

These relations are read as

$L(x, y, z)$: point number x is to the left of line from point number y to point number z.

$R(x, y, z)$: point number x is to the right of line from point number y to point number z.

The axioms are as follows

AS1. $L(x, y, z) \rightarrow \sim x = y$,
$L(x, y, z) \rightarrow \sim x = z$,
$R(x, y, z) \rightarrow \sim x = y$,
$R(x, y, z) \rightarrow \sim x = z$.

AS2. $L(x, y, z) \ \&\ \sim x = v \rightarrow L(v, x, y) \vee R(v, x, y) \vee L(v, x, z) \vee R(v, x, z)$,
$R(x, y, z) \ \&\ \sim x = v \rightarrow L(v, x, y) \vee R(v, x, y) \vee L(v, x, z) \vee R(v, x, z)$.

AS3. $L(x, y, z) \rightarrow L(y, z, x)$,
$R(x, y, z) \rightarrow R(y, z, x)$,
$L(x, y, z) \rightarrow L(x, z, y)$,
$R(x, y, z) \rightarrow L(x, z, y)$.

AS4. $\sim (L(x, y, z) \ \&\ R(x, y, z))$.

AS1 and AS2 are mere rewritings of the geometrical axioms and principles. AS3 determines all the six configurations for three points, given one of them like $L(x, y, z)$. AS4 is the essential principle for deriving Sylvester's conjecture for more than six points.

Let us define apartness as in the geometrical case:

Definition 18. $A(x, y, z) =_{df} L(x, y, z) \vee R(x, y, z)$.

Because $A(x, y, z) \rightarrow A(x, z, y)$, we can have the convention that $y < z$ in $A(x, y, z)$. *Sylvester's hypothesis in arithmetic form*, for n points, now is

$ASH(n)$: $A(1|\ldots|n, x, y)$

The conjecture is, with the last two arguments ordered as in proof of theorem 4,

$ASC(n)$: $A(3 \cdot \ldots \cdot n, 1, 2) \vee (A(2, 1, 3)\ \&\ A(4 \cdot \ldots \cdot n, 1, 3)) \vee \ldots \vee A(1 \cdot \ldots \cdot n-2, n-1, n)$.

Theorem 19. ASH(n) →ASC(n) *is algorithmically decidable for any given n.*

Proof. Our translation uses only propositional logic with free variable atomic formulas, therefore it is decidable whether $ASH(n) \rightarrow ASC(n)$ is constructively provable from AS1–AS4.

It is straightforward to give a translation of the axioms in Sec. 2, where we did not have the $L-R$ ordering principles. Independence of the seven-point Sylvester conjecture in constructive incidence geometry can now be shown by purely syntactic, proof-theoretical methods, as opposed to earlier independence proofs based on models.

[Dijkstra 1989, p. 223] suggests that Sylvester's conjecture for the Euclidean plane "is a truly geometrical theorem and not a combinatorial one," but our translation shows that this is not the case.

9 Concluding remarks

The first proofs of Sylvester's conjecture used the Euclidean notion of distance, where a minimum condition identified the two-point line [see Coxeter 1989]. Such conditions do not give constructive existence proofs, and a similar remark applies to other previous approaches to the conjecture [see, e.g., Kelly and Moser 1958]. Our constructive existence proof gives a method for effectively finding a two-point line from purely geometric data of the form $A(a, l)$. Implementation of geometric algorithms is described in [von Plato 1995]. All results of that work have been implemented in the Coq proof editor by Kahn [Kahn 1995].

The most obvious further question would be to give a general inductive proof of Sylvester's conjecture from a natural set of axioms. In another direction, we can formulate analogies to Sylvester's conjecture and its dual in geometries of higher dimensions, for example: Given n distinct, concurrent and *nonplanar* lines $l_1, ..., l_n$, there is at least one plane with just two of the lines incident with it. Dual: Given n *inclined* planes such that at least one line of intersection is apart from at least one plane, there is a line of intersection incident with just two of the planes.

We defined betweenness by the relation Div that is symmetric with respect to the two points. Div itself was defined in terms of L and R. It is possible to define a two-place strict order relation for the points of a line in terms of the basic concepts of the oriented affine plane. Betweenness, in turn, is definable from a strict order. We needed only the symmetrized relation for the present purposes.

In a full treatment of the ordered affine plane, the notion of convergence has two aspects. It is the conjunction of the constructive version of avoidance of equal and opposite directedness; It is also refined into the disjunction of two kinds of intersection: Lines are directed, so that line l can intersect line m from left to right or right to left. With the concepts of equally and oppositely directed lines, line equality and parallelism both have two cases. By the use of constructive versions of these concepts (unequally and inoppositely directed lines), it is also possible to give a stronger formulation of the substitution axiom OIVb, analogously to axiom OIVa. Then the triangle axioms for directed lines, as well as the ordering axioms for parallel lines, OV and AF5 above, come out as theorems. A constructive theory to this effect has been presented in [von Plato 1998]. A drawback, in comparison to the present approach, is the higher number of basic concepts that calls for many more axioms.

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