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Perhaps the Intermediate Value Theorem[1]

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Abstract: In the context of intuitionistic real analysis, we introduce the set $\mathcal{F}$ consisting of all continuous functions ϕ from $[0, 1]$ to $\mathbb{R}$ such that $\phi(0) = 0$ and $\phi(1) = 1$. We let $\mathcal{I}_0$ be the set of all ϕ in $\mathcal{F}$ for which we may find x in $[0, 1]$ such that $\phi(x) = \frac{1}{2}$. It is well-known that there are functions in $\mathcal{F}$ that we can not prove to belong to $\mathcal{I}_0$, and that, with the help of Brouwer's Continuity Principle one may derive a contradiction from the assumption that $\mathcal{I}_0$ coincides with $\mathcal{F}$. We show that Brouwer's Continuity Principle also enables us to define uncountably many subsets $\mathcal{G}$ of $\mathcal{F}$ with the property $\mathcal{I}_0 \subseteq \mathcal{G} \subset (\mathcal{I}_0)^{\neg\neg}$.

Key Words: Intuitionistic real analysis, intermediate value theorem, perhaps
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1 Introduction

We let $\mathcal{F}$ be the set of all continuous functions ϕ from $[0, 1]$ to $\mathbb{R}$ such that $\phi(0) = 0$ and $\phi(1) = 1$. We let $\mathcal{I}_0$ be the set of all functions ϕ in $\mathcal{F}$ that assume the value $\frac{1}{2}$, that is, there exists a number x in $[0, 1]$ such that $\phi(x) = \frac{1}{2}$. If the statement "there exists a number x in $[0, 1]$ such that $\phi(x) = \frac{1}{2}$" be constructively true, as we intend it to be, we must be able to approximate a number with the promised property with any degree of accuracy, and, therefore, for every n, we must be able to find rational numbers p, q in $[0, 1]$ such that $p < q$ and $q - p < \frac{1}{2^n}$ and there exists x in $[p, q]$ such that $\phi(x) = \frac{1}{2}$.

Under this very natural constructive interpretation the classical Intermediate Value Theorem, that is, the statement that the sets $\mathcal{I}_0$ and $\mathcal{F}$ coincide, is false: sometimes, the point where a given function in $\mathcal{F}$ would assume the value $\frac{1}{2}$ can not be located and one does not have the slightest idea where it might be found. When asked to provide an example to make this clear, the constructive mathematician may give and often does give something like the following answer:

> Consider the class $\mathcal{J}_1$ consisting of all functions ϕ in $\mathcal{F}$ such that ϕ is linear on $[0, \frac{1}{3}]$ and on $[\frac{2}{3}, 1]$, and constant on $[\frac{1}{3}, \frac{2}{3}]$.
>
> Suppose that ϕ belongs to the class $\mathcal{J}_1$. Observe that ϕ is completely determined once we have chosen $y := \phi(\frac{1}{3})$. Note that, if $y > \frac{1}{2}$, then ϕ assumes the value $\frac{1}{2}$ at the point $\frac{1}{6y}$, and at this point only, and this

[1] C. S. Calude, H. Ishihara (eds.). *Constructivity, Computability, and Logic. A Collection of Papers in Honour of the 60th Birthday of Douglas Bridges.*

point is smaller than $\frac{1}{3}$, and if $y < \frac{1}{2}$, then ϕ assumes the value $\frac{1}{2}$ at the point $x = \frac{2}{3} + \frac{\frac{1}{2}-y}{3(1-y)}$, and at this point only, and this point is greater than $\frac{2}{3}$. If $y = \frac{1}{2}$, then ϕ assumes the value $\frac{1}{2}$ at every point x in $[\frac{1}{3}, \frac{2}{3}]$.

Now suppose that ϕ belongs to the class $\mathcal{J}_1$ and that we find x in $[0,1]$ such that $\phi(x) = \frac{1}{2}$. By making a first approximation of the number x we must be able to prove either $x < \frac{2}{3}$ or $x > \frac{1}{3}$. Observe that, if $x < \frac{2}{3}$ then $y = \phi(\frac{1}{2}) \geq \frac{1}{2}$ and if $x > \frac{1}{3}$ then $y \leq \frac{1}{2}$.

If we now choose a number y for which we are unable to decide $y \geq \frac{1}{2}$ or $y \leq \frac{1}{2}$, and make $\phi(\frac{1}{3}) = y$, we will be unable, for the corresponding function ϕ, to indicate x such that $\phi(x) = \frac{1}{2}$.

There do exist real numbers y for which we are unable to decide $y \geq \frac{1}{2}$ or $y \leq \frac{1}{2}$, as appears from the following *example in Brouwer's style*:

Consider $y := \lim_{n\to\infty} y_n$, where, for each n, $y_n = \frac{1}{2} + (-1)^n \frac{1}{n}$, if, in the first n digits of the decimal expansion of π, there does not occur an uninterrupted seqence of 99 9's, and $y_n = y_{n-1}$, if there does.

We have no proof that $y \geq \frac{1}{2}$: the statement $y \geq \frac{1}{2}$ implies that, if there exists an uninterrupted sequence of 99 9's in the decimal expansion of π, then the first such sequence will be concluded at an odd place in the expansion, and we have no knowledge of this fact.

We also have no proof of the statement $y \leq \frac{1}{2}$.

The statement that, for every real number y, either $y \leq \frac{1}{2}$ or $y \geq \frac{1}{2}$, is equivalent to the principle *LLPO*, the *Lesser Limited Principle of Omniscience*, see [1]. Clearly, this 'principle', a weakening of the principle of the excluded middle, is wrong, if one, as Brouwer proposed, interprets the logical constants constructively.

If one nevertheless should accept *LLPO* as an axiom, (in a perhaps nostalgic but not well-founded attempt to develop a kind of mathematics in between classical and intuitionistic mathematics), the Intermediate Value Theorem, that is, the statement that the sets $\mathcal{F}$ and $\mathcal{I}_0$ coincide, may be 'proved' by the method of successive bisection, as is noticed in [1]. Of course, the idea to promote *LLPO* to the status of an axiom is far from the mind of the constructive mathematician.

This paper finds its origin in the observation that for any function ϕ in the class $\mathcal{J}_1$ the following holds true:

If $\phi(\frac{1}{2}) \# \frac{1}{2}$, then there exists x in $[0,1]$ such that $\phi(x) = \frac{1}{2}$.

The symbol # denotes the constructive *(positive) inequality* or *apartness relation* on the real numbers. For real numbers x, y, $x \# y$ if and only if we are able to calculate n such that $|x - y| > \frac{1}{2^n}$.

We now define, for any function ϕ in $\mathcal{F}$: ϕ *perhaps assumes the value* $\frac{1}{2}$ if and only if there exists x in $[0,1]$ such that, if $\phi(x)\#\frac{1}{2}$, then ϕ assumes the value $\frac{1}{2}$, that is, as soon as we find evidence that x positively fails to be a point where ϕ assumes the value $\frac{1}{2}$, we will be able to calculate a number z in $[0,1]$ with the property $\phi(z) = \frac{1}{2}$.

We want the reader to understand the word "perhaps" as an expression of caution. The statement: "ϕ perhaps assumes the value $\frac{1}{2}$" is not as full a promise as the statement: "ϕ assumes the value $\frac{1}{2}$". More precisely, if I say: "ϕ perhaps assumes the value $\frac{1}{2}$" I give you a number and you should try for yourself if ϕ assumes the value $\frac{1}{2}$ at the given point. In case you discover that the number I gave you is no good, because the function ϕ assumes at that point a value apart from $\frac{1}{2}$, you may come back and will be given a number at which the function surely assumes the value $\frac{1}{2}$.

The class of all functions in $\mathcal{F}$ that perhaps assume the value $\frac{1}{2}$ will be called $\mathcal{I}_1$.

We just observed that every function in $\mathcal{J}_1$ belongs to $\mathcal{I}_1$.

We now may ask: do the sets $\mathcal{F}$ and $\mathcal{I}_1$ coincide or do there exist functions ϕ in the set $\mathcal{F}$ for which we can not even prove that they perhaps assume the value $\frac{1}{2}$? It turns out that the sets $\mathcal{F}$ and $\mathcal{I}_1$ do not coincide and that there are functions in $\mathcal{F}$ for which we can not prove that they belong to $\mathcal{I}_1$, as appears from the following.

> Consider the class $\mathcal{J}_2$ consisting of all continuous functions ϕ from $[0,1]$ to $\mathbb{R}$, such that $\phi(0) = 0$ and $\phi(1) = 1$ and ϕ is linear on $[0, \frac{1}{5}]$, on $[\frac{2}{5}, \frac{3}{5}]$ and on $[\frac{4}{5}, 1]$, and ϕ is constant on $[\frac{1}{5}, \frac{2}{5}]$, and also on $[\frac{3}{5}, \frac{4}{5}]$.
>
> Any member of the class $\mathcal{J}_2$ is determined once we have chosen $y_0 = \phi(\frac{1}{5})$ and $y_1 = \phi(\frac{3}{5})$.
>
> Let us choose such y_0 and y_1 and let us consider the corresponding function ϕ.
>
> Assume that ϕ perhaps assumes the value $\frac{1}{2}$. We then determine x in $[0,1]$ such that, if $\phi(x)\#\frac{1}{2}$, then there exists y in $[0,1]$ with the property $\phi(y) = \frac{1}{2}$, and we distinguish two cases:
> *Case (i).* $x < \frac{3}{5}$. Suppose that $y_0 < \frac{1}{2}$. In that case, $\phi(x)\#\frac{1}{2}$, and there exists z such that $\phi(z) = \frac{1}{2}$, and, therefore, either $y_1 \leq \frac{1}{2}$ or $y_1 \leq \frac{1}{2}$, by the argument that we used when discussing the class $\mathcal{J}_1$.
> *Case (ii).* $x > \frac{2}{5}$. Suppose that $y_1 > \frac{1}{2}$. In that case, $\phi(x)\#\frac{1}{2}$, and there exists z such that $\phi(z) = \frac{1}{2}$, and, therefore, either $y_0 \leq \frac{1}{2}$ or $y_0 \leq \frac{1}{2}$, again by the argument that we used when discussing the class $\mathcal{J}_1$.
>
> We thus may conclude: *Either: if* $y_0 < \frac{1}{2}$, *then* $y_1 \leq \frac{1}{2}$ *or* $y_1 \leq \frac{1}{2}$, *or: if* $y_1 > \frac{1}{2}$, *then* $y_0 \leq \frac{1}{2}$ *or* $y_0 \leq \frac{1}{2}$.

The latter statement would follow from the (wrong) principle $LLPO$, but it seems somewhat weaker. Nevertheless, it is false as well as $LLPO$ itself. In order to see this, the reader should study the following example:

> Let y_0 be the number y we defined a moment ago, and let y_1 be defined similarly: take the definition of y and replace "99 9's" by "88 8's".

The reader may also verify that, for any function ϕ in the class $\mathcal{J}_2$,

> if $\phi(\frac{1}{2})\#\frac{1}{2}$, then *either:* $\phi(\frac{1}{2}) < \frac{1}{2}$ and, if $\phi(\frac{7}{10})\#\frac{1}{2}$, then ϕ assumes the value $\frac{1}{2}$, *or:* $\phi(\frac{1}{2}) > \frac{1}{2}$ and, if $\phi(\frac{3}{10})\#\frac{1}{2}$, then ϕ assumes the value $\frac{1}{2}$, and, therefore, if $\phi(\frac{1}{2})\#\frac{1}{2}$, then ϕ perhaps assumes the value $\frac{1}{2}$.

We now define, for any function ϕ in $\mathcal{F}$: ϕ *perhaps perhaps assumes the value* $\frac{1}{2}$ if and only if there exists x in $[0, 1]$ such that, if $\phi(x)\#\frac{1}{2}$, then ϕ perhaps assumes the value $\frac{1}{2}$. The class of all functions in $\mathcal{F}$ that perhaps perhaps assume the value $\frac{1}{2}$ will be called $\mathcal{I}_2$.

Any function from the class $\mathcal{J}_2$ belongs to the class $\mathcal{I}_2$.

It will (perhaps) be clear now what we want to do: we want to iterate "perhaps" and prove that we obtain larger and larger classes of functions. This result may be taken as evidence for the great expressivity of the language of intuitionistic mathematics.

The phenomenon of "perhapsity" also occurs in other contexts. It is useful to mention two more examples. More information on these examples may be found in [5], [6], and [9].

Let A be a decidable subset of the set $\mathbb{N}$ of the natural numbers. (We call a subset A of $\mathbb{N}$ a *decidable* subset of $\mathbb{N}$ if and only if there exists α in $\mathcal{N}$ such that, for every n, $n \in A$ if and only if $\alpha(n) = 1$. We do not require that the function α is given by a finite algorithm). A is *finite* if and only if there exists n such that, for all $m > n$, m does not belong to A. A is *perhaps-finite* if and only if there exists n such that, for all $m > n$, if m belongs to A, then A is finite.

Let x be a real number. x is *rational* if and only if there exists a rational number q such that x coincides with q. x is *perhaps-rational* if and only if there exists a rational number q such that, if $x\#q$, then x is rational.

The further contents of this paper are as follows. In Section 2, we introduce Brouwer's Continuity Principle. In Section 3, we prove that iterating "perhaps" finitely many times gives rise to larger and larger sets of functions that we call *perhapsive extensions of* $\mathcal{I}_0$. In Section 4, we prove that many infinite sequences of perhapsive extensions of $\mathcal{I}_0$ have an upper bound, that may be used to obtain further perhapsive extensions. In Section 5, we introduce stumps, the intuitionistic substitute for classical countable ordinals, and use them to label perhapsive extensions of $\mathcal{I}_0$. In Section 6 we make some concluding remarks.

2 Brouwer's Continuity Principle

We are contributing to intuitionistic analysis and adhere to the constructive interpretation of the logical constants. In particular, a disjunctive statement $A \vee B$ is considered proven if and only if we either have a proof of A or a proof of B, and a proof of an existential statement $\exists x \in V[A(x)]$ should provide one with an element x_0 of the set V and a proof of $A(x_0)$.

We let $\mathbb{N}$ be the set of the natural numbers and use the letters $m, n, \dots$ as variables over this set.

We let $\mathcal{N}$ be the set of all infinite sequences of natural numbers, that is, the set of all functions from $\mathbb{N}$ to $\mathbb{N}$. We use $\alpha, \beta, \dots$ as variables over the set $\mathcal{N}$.

For every α, for every n, we denote the value that α assumes at n by $\alpha(n)$.

For every α, for every n, we denote the finite sequence $\alpha(0), \alpha(1), \dots, \alpha(n-1)$ by $\overline{\alpha}n$ or $\overline{\alpha}(n)$.

The following principle was sometimes used by Brouwer and is an axiom of intuitionistic analysis.

Brouwer's Continuity Principle:

> *For every binary relation $R \subseteq \mathcal{N} \times \mathbb{N}$, if for every α there exists m such that $\alpha R m$, then for every α there exist m, n such that for every β, if $\overline{\alpha}n = \overline{\beta}n$, then $\beta R m$.*

If one wants to understand why this axiom is judged to be plausible, one should think of the fact that, for the intuitionistic mathematician, an infinite sequence $\alpha = \alpha(0), \alpha(1), \alpha(2), \dots$ of natural numbers is not necessarily given by means of the description of a method to find its values, but may be the result of a free step-by-step-construction, and thus as a growing object that is always incomplete and, in some sense, given by a black box.

Suppose now we are able to calculate, for every infinite sequence α of natural numbers, a natural number m suitable for α, that is, such that $\alpha R m$. We are interpreting both the "for every α" and to the "there exists" seriously, that is, constructively: given *any* infinite sequence *whatsoever* from the very large and unsurveyable set $\mathcal{N}$ we know how to *effectively* discover a natural number suitable for it. In particular we must be able to find a suitable number if the sequence is being created step by step. A number m, suitable for an α that is given step by step will be discovered and recognized as such at some moment of time, and at that moment only finitely many values of α, say $\alpha(0), \alpha(1), \dots, \alpha(n-1)$ will be known. The number m will therefore suit every β that has its first n values the same as α. One should not think that the number m is only suitable for sequences that are created step by step. The intuitionistic mathematician believes that also a sequence that obeys a rule that admits of a finite description might be the result of a step-by-step-construction. One should not exclude the possibility that a 'black box' has as its output the decimal expansion of π.

Brouwer's Continuity Principle is a bold assertion, inspired by some negative mathematical experiences, as one may learn from studying [2]. Such a negative experience is, for instance, the fact that a real function that has the value 0 at every point $x < 0$ and the value 1 at every point $x \geq 0$ cannot be considered to be defined everywhere. So what does it mean to have positive evidence that the function is defined everywhere?

Brouwer's Continuity Principle has many consequences that are very surprising from a classical, that is: non-intuitionistic, point of view, see [7]. The result of this paper is another illustration of this fact.

3 The first countably many perhapsive extensions of $\mathcal{I}_0$

3.1 The first perhapsive extension

Let $\mathcal{G}$ be a subset of $\mathcal{F}$, such that $\mathcal{I}_0$ is a subset of $\mathcal{G}$. We let $\mathcal{G}^+$ be the set of all functions ϕ in $\mathcal{F}$ such that, for some x in $[0,1]$, if $\phi(x)\#\frac{1}{2}$, then ϕ belongs to $\mathcal{G}$, and we call the set $\mathcal{G}^+$ *the first perhapsive extension of* $\mathcal{G}$.

A subset $\mathcal{G}$ of $\mathcal{F}$ containing $\mathcal{I}_0$ will be called *perhapsive* if and only if $\mathcal{G}^+$ coincides with $\mathcal{G}$.

Note that $\mathcal{I}_1$ coincides with $(\mathcal{I}_0)^+$ and that $\mathcal{I}_2$ coincides with $(\mathcal{I}_1)^+$.

For every subclass $\mathcal{V}$ of $\mathcal{F}$ we let $\mathcal{V}^\neg$, *the complement of* $\mathcal{V}$ *in* $\mathcal{F}$, be the set of all ϕ in $\mathcal{F}$ that do not belong to $\mathcal{V}$, where the word "not" is used in the strong (and usual) sense that the assumption that ϕ belongs to $\mathcal{V}$ leads to a contradiction. It is a fact, well-known from intuitionistic logic, that, for every subclass $\mathcal{V}$ of $\mathcal{F}$, $\mathcal{V}^{\neg\neg\neg}$ coincides with $\mathcal{V}^\neg$.

Lemma 1. *(i) For every subclass $\mathcal{G}$ of $\mathcal{F}$ containing $\mathcal{I}_0$, $\mathcal{G}$ is a subclass of $(\mathcal{G})^+$.*

(ii) For every subclass $\mathcal{G}$ of $\mathcal{F}$ containing $\mathcal{I}_0$, if $\mathcal{G}$ is a subclass of $(\mathcal{I}_0)^{\neg\neg}$, then $\mathcal{G}^+$ is a subclass of $(\mathcal{I}_0)^{\neg\neg}$.

(iii) The set $(\mathcal{I}_0)^{\neg\neg}$ is perhapsive.

Proof. (i) The proof is left to the reader.

(ii)Assume that ϕ belongs to $\mathcal{G}^+$. Find x in $[0,1]$ such that, if $\phi(x)\#\frac{1}{2}$, then ϕ belongs to $\mathcal{G}$ and distinguish two cases.

First Case. $\phi(x)\#\frac{1}{2}$. Then ϕ belongs to $\mathcal{G}$ and thus to $(\mathcal{I}_0)^{\neg\neg}$.

Second Case. $\phi(x) = \frac{1}{2}$. Then ϕ belongs to $\mathcal{I}_0$ and thus to $(\mathcal{I}_0)^{\neg\neg}$.

Now observe that $\neg\neg(\phi(x)\#\frac{1}{2}$ or $\phi(x) = \frac{1}{2})$. Therefore $\neg\neg(\phi$ belongs to $(\mathcal{I}_0)^{\neg\neg})$, and thus ϕ belongs to $(\mathcal{I}_0)^{\neg\neg\neg\neg}$ and, therefore, also to $(\mathcal{I}_0)^{\neg\neg}$.

(iii) The proof is left to the reader. □

3.2 An invariance property

We shall make use of the fact that the class $\mathcal{I}_0$ and the classes one obtains from $\mathcal{I}_0$ by repeatedly forming perhapsive extensions, enjoy a certain invariance property. We first define this property.

Let a, b be elements of $[0,1]$ such that $a < b$ and let ρ be an element of the class $\mathcal{F}$.

For every element ψ of $\mathcal{F}$ we let $G_{a,b,\rho}[\psi]$ be the element of $\mathcal{F}$ such that

(i) for every x in $[0,a] \cup [b,1]$, $G_{a,b,\rho}[\psi](x) = \rho(x)$.

(ii) for every x in $[a,b]$, $G_{a,b,\rho}[\psi](x) = \rho(a) + (\rho(b) - \rho(a))\psi(x)$

Now let $\mathcal{G}$ be a subclass of $\mathcal{F}$. We say that $\mathcal{G}$ *has the invariance property* if and only if, for all a, b in $[0,1]$ such that $a < b$, for every ρ in $\mathcal{F}$ such that for every x in $[0,a]$, for every y in $[b,1]$, $\rho(x) < \frac{1}{2} < \rho(y)$ and $\rho(a) + \rho(b) = 1$, for every ψ in $\mathcal{F}$, ψ belongs to $\mathcal{G}$ if and only if $G_{a,b,\rho}[\psi]$ belongs to $\mathcal{G}$.

Lemma 2. *(i) The class $\mathcal{I}_0$ has the invariance property.*

(ii) For every subclass $\mathcal{G}$ of $\mathcal{F}$, if $\mathcal{G}$ has the invariance property, then $\mathcal{G}^+$ has the invariance property.

Proof. Assume that a, b are elements of $[0,1]$, such that $a < b$, and that ρ is an element of $\mathcal{F}$ such that for every x in $[0,a]$, for every y in $[b,1]$, $\rho(x) < \frac{1}{2} < \rho(y)$ and $\rho(a) + \rho(b) = 1$, and let ψ be an element of $\mathcal{F}$.

(i) Suppose that ψ belongs to $\mathcal{I}_0$. Find x such that $\phi(x) = \frac{1}{2}$, and note that $G_{a,b,\rho}[\psi](a + (b-a)x) = \frac{1}{2}$, and, therefore, $G_{a,b,\rho}[\psi]$ belongs to $\mathcal{I}_0$.

Suppose that $G_{a,b,\rho}[\psi]$ belongs to $\mathcal{I}_0$. Find x such that $G_{a,b,\rho}[\psi](x) = \frac{1}{2}$ and note that x belongs to $[a,b]$. Therefore, $\psi(\frac{x-a}{b-a}) = \frac{1}{2}$ and ψ belongs to $\mathcal{I}_0$.

Now assume in addition that $\mathcal{G}$ is a subclass of $\mathcal{F}$ enjoying the invariance property.

(ii) Suppose that ψ belongs to $\mathcal{G}^+$. Find x in $[0,1]$ such that, if $\psi(x)\#\frac{1}{2}$, then ψ belongs to $\mathcal{G}$. Note that, if $G_{a,b,\rho}[\psi](a+(b-a)x)\#\frac{1}{2}$, then $\psi(x)\#\frac{1}{2}$, and, therefore, ψ belongs to $\mathcal{G}$, and, by the additional assumption, $G_{a,b,\rho}[\psi]$ belongs to $\mathcal{G}$. Clearly, $G_{a,b,\rho}[\psi]$ belongs to $\mathcal{G}^+$.

Suppose that $G_{a,b,\rho}[\psi]$ belongs to $\mathcal{G}^+$. Find x in $[0,1]$ such that, if $\psi(x)\#\frac{1}{2}$, then $G_{a,b,\rho}[\psi]$ belongs to $\mathcal{G}$. Note that, for every y in $[0,a]$, for every z in $[b,1]$, $\rho(y) < \frac{1}{2} < \rho(z)$ and that ρ is continuous at the points a, b, and find m such that, for every y in $[0, a + \frac{1}{2^m}] \cup [b - \frac{1}{2^m}, 1]$, $\phi(y)\#\frac{1}{2}$. Now *either* x belongs to $[0, a + \frac{1}{2^m}] \cup [b - \frac{1}{2^m}, 1]$ *or* x belongs to $[a,b]$, and

we may distinguish two cases.

Case (a). $\phi(x)\#\frac{1}{2}$. Now, $G_{a,b,\rho}[\psi]$ belongs to $\mathcal{G}$, and by the additional assumption, also ψ belongs to $\mathcal{G}$ and thus to $\mathcal{G}^+$.

Case (b). x belongs to $[a,b]$. Note that, if $\psi(\frac{x-a}{b-a})\#\frac{1}{2}$, then $G_{a,b,\rho}[\psi](x)\#\frac{1}{2}$, and, therefore, $G_{a,b,\rho}[\psi]$ belongs to $\mathcal{G}$, and, by the additional assumption, ψ belongs to $\mathcal{G}$. Clearly, ψ belongs to $\mathcal{G}^+$. □

3.3 $\mathcal{I}_0$ is not perhapsive

The first thing that we want to prove is the fact that, in the presence of Brouwer's Continuity Principle, the assumption that $\mathcal{I}_0$ is perhapsive, (that is: $(\mathcal{I}_0)^+$ coincides with $\mathcal{I}_0$), leads to a contradiction. This is of course no different from proving that Brouwer's Continuity Principle is incompatible with $LLPO$, but we prove the result in a form that will turn out to be useful in the sequel.

We take some preparatory steps.

Let $\mathbb{N}^*$ be the set of all finite sequences of natural numbers. We use the symbol $*$ to denote the operation of concatenating finite sequences of natural numbers, that is, for all s, t in $\mathbb{N}^*$, $s * t$ denotes the finite sequence one obtains by putting t behind s.

For every s in $\mathbb{N}^*$, for every α, we define: α *passes through* s if and only if there exists n such that $\overline{\alpha}n = s$.

For every natural number n we let $\underline{n}$ be the element of $\mathcal{N}$ with the constant value n, that is, for all i, $\underline{n}(i) = n$.

For every s in $\mathbb{N}$ we define an element ϕ_s of $\mathcal{F}$, as follows.

(i) For all n, $\phi_{\overline{\underline{0}}n}$ is an element of $\mathcal{F}$ that is linear on each of the segments $[0, \frac{1}{3}]$, $[\frac{1}{3}, \frac{2}{3}]$ and $[\frac{2}{3}, 1]$ and takes the value $\frac{1}{2}$ at the points $\frac{1}{3}$ and $\frac{2}{3}$.

(ii) For all n, p in $\mathbb{N}$, for all s in $\mathbb{N}^*$, $\phi_{\overline{\underline{0}}n*\langle 2p+1\rangle *s}$ is an element of $\mathcal{F}$ that is linear on each of the segments $[0, \frac{1}{3}]$, $[\frac{1}{3}, \frac{2}{3}]$ and $[\frac{2}{3}, 1]$ and takes the value $\frac{1}{2} + \frac{1}{2^n}$ at the points $\frac{1}{3}$ and $\frac{2}{3}$.

(iii) For all n, p in $\mathbb{N}$, for all s in $\mathbb{N}^*$, $\phi_{\overline{\underline{0}}n*\langle 2p+2\rangle *s}$ is an element of $\mathcal{F}$ that is linear on each of the segments $[0, \frac{1}{3}]$, $[\frac{1}{3}, \frac{2}{3}]$ and $[\frac{2}{3}, 1]$ and takes the value $\frac{1}{2} - \frac{1}{2^n}$ at the points $\frac{1}{3}$ and $\frac{2}{3}$.

For every α in $\mathcal{N}$ we define an element ϕ_α of $\mathcal{F}$ by: for every x in $[0, 1]$, $\phi_\alpha(x) = \lim_{n\to\infty} \phi_{\overline{\alpha}n}(x)$.

Note that, for each α, ϕ_α is well-defined and belongs to $\mathcal{J}_1$.

Lemma 3. *(i) For each α, ϕ_α belongs to $(\mathcal{I}_0)^+$.*

(ii) The assumption that, for each α, ϕ_α belongs to $\mathcal{I}_0$, leads to a contradiction.

Proof. (i) Let α belong to $\mathcal{N}$ and assume $\phi_\alpha(\frac{1}{2})\#\frac{1}{2}$. Find m such that $|\phi_\alpha(\frac{1}{2}) - \frac{1}{2}| > \frac{1}{2^m}$ and then find n, p such that α passes through either $\overline{\underline{0}}n * \langle 2p + 2\rangle$ or $\overline{\underline{0}}n * \langle 2p + 1\rangle$. If α passes through $\overline{\underline{0}}n * \langle 2p + 2\rangle$, then ϕ_α assumes the value $\frac{1}{2}$ at the point $x = \frac{2}{3} + \frac{1}{3(2^n+1)}$, and if α passes through $\overline{\underline{0}}n * \langle 2p+1\rangle$, then ϕ_α assumes the value $\frac{1}{2}$ at the point $x = \frac{1}{3(1+2^{n-1})}$. Thus we see that, if $\phi_\alpha(\frac{1}{2}) \# \frac{1}{2}$, then ϕ_α assumes the value $\frac{1}{2}$ and ϕ_α belongs to $\mathcal{I}_0$. Therefore, for each α, ϕ_α belongs to $(\mathcal{I}_0)^+$.

(ii) Assume that, for each α, ϕ_α belongs to $\mathcal{I}_0$. Then, for each α, we may determine $i < 2$ such that, if $i = 0$, then there exists x in $[0, \frac{2}{3}]$ such that $\phi_\alpha(x) = \frac{1}{2}$, and, if $i = 1$, then there exists x in $[\frac{1}{3}, 1]$ such that $\phi_\alpha(x) = \frac{1}{2}$. Applying Brouwer's Continuity Principle, we find n, i such that, *either* $i = 0$ and, for every α, if $\overline{\alpha}n = \overline{\underline{0}}n$, then there exists x in $[0, \frac{2}{3}]$ such that $\phi_\alpha(x) = \frac{1}{2}$, *or* $i = 1$ and, for every α, if $\overline{\alpha}n = \overline{\underline{0}}n$, then there exists x in $[\frac{1}{3}, 1]$ such that $\phi_\alpha(x) = \frac{1}{2}$. It follows that *either* for every α, if $\overline{\alpha}n = \overline{\underline{0}}n$, then, for each p, if p is the least n such that $\alpha(n) \neq 0$, then $\alpha(p)$ is odd, or for every α, if $\overline{\alpha}n = \overline{\underline{0}}n$, then, for each p, if p is the least n such that $\alpha(n) \neq 0$, then $\alpha(p)$ is even. Both alternatives are wrong, as we see from the examples $\overline{\underline{0}}(n) * \langle 2 \rangle * \underline{0}$ and $\overline{\underline{0}}(n) * \langle 1 \rangle * \underline{0}$. □

3.4 Positive nonperhapsity

Let $\mathcal{G}$ be a subset of $\mathcal{F}$ containing $\mathcal{I}_0$. We call the set $\mathcal{G}$ *positively nonperhapsive* if and only if

> there exists a mapping $s \mapsto \psi_s$ from $\mathbb{N}^*$ to $\mathcal{F}$ such that, for every α in $\mathcal{N}$, for every x in $[0,1]$, $\lim_{n\to\infty} \psi_{\overline{\alpha}n}(x)$ exists, and, if one defines a mapping $\alpha \mapsto \psi_\alpha$ from $\mathcal{N}$ to $\mathcal{F}$ by: for each α, for each x in $[0,1]$, $\psi_\alpha(x) = \lim_{n\to\infty} \psi_{\overline{\alpha}n}(x)$, then, for every α, ψ_α belongs to $\mathcal{G}^+$ and not for every α, ψ_α belongs to $\mathcal{G}$.

We have just seen that $\mathcal{I}_0$ itself is positively nonperhapsive. We want to show that also $\mathcal{I}_1$ is positively nonperhapsive. Actually, we shall prove that the first perhapsive extension of a positively nonperhapsive set is itself positively nonperhapsive.

Suppose that we defined, for each s in $\mathbb{N}^*$, an element ψ_s of $\mathcal{F}$, in such a way that, for every α in $\mathcal{N}$, for every x in $[0,1]$, $\lim_{n\to\infty} \psi_{\overline{\alpha}n}(x)$ exists. We define, for each s in $\mathbb{N}^*$, another element of $\mathcal{F}$ that we want to call ψ_s^+.

> (i) For all n, $\psi^+_{\overline{\underline{0}}n}$ is an element of $\mathcal{F}$ that is linear on each of the segments $[0, \frac{1}{5}]$, $[\frac{1}{5}, \frac{4}{5}]$ and $[\frac{4}{5}, 1]$ and takes the value $\frac{1}{2}$ at the point $\frac{1}{5}$ and at the point $\frac{4}{5}$.
>
> (ii) For all n, p in $\mathbb{N}$, for all s in $\mathbb{N}^*$, $\psi^+_{\overline{\underline{0}}n*\langle 2p+1\rangle * s}$ is an element of $\mathcal{F}$ that is linear on the segments $[0, \frac{1}{5}]$, $[\frac{2}{5}, \frac{4}{5}]$ and $[\frac{4}{5}, 1]$, takes the value $\frac{1}{2} + \frac{1}{2^n}$ at the points $\frac{2}{5}$ and $\frac{4}{5}$, and the value $\frac{1}{2} - \frac{1}{2^n}$ at $\frac{1}{5}$ and satisfies: for all x in $[\frac{1}{5}, \frac{2}{5}]$, $\psi^+_{\overline{\underline{0}}n*\langle 2p+1\rangle * s}(x) = \frac{1}{2} + \frac{1}{2^{n-1}}(\psi_s(5x-1) - \frac{1}{2})$.

(iii) For all n, p in $\mathbb{N}$, for all s in $\mathbb{N}^*$, $\psi^+_{\underline{0}n*\langle 2p+2\rangle*s}$ is an element of $\mathcal{F}$ that is linear on the segments $[0, \frac{1}{5}]$, $[\frac{1}{5}, \frac{3}{5}]$ and $[\frac{4}{5}, 1]$, takes the value $\frac{1}{2} - \frac{1}{2^n}$ at the points $\frac{1}{5}$ and $\frac{3}{5}$, and the value $\frac{1}{2} + \frac{1}{2^n}$ at $\frac{4}{5}$ and satisfies: for all x in $[\frac{3}{5}, \frac{4}{5}]$, $\psi^+_{\underline{0}n*\langle 2p+2\rangle*s}(x) = \frac{1}{2} + \frac{1}{2^{n-1}}(\psi_s(5x-3) - \frac{1}{2})$.

Note that, for every α, $\lim_{n\to\infty} \psi^+_{\overline{\alpha}n}(x)$ exists. We thus may define a mapping $\alpha \mapsto \psi^+_\alpha$ from $\mathcal{N}$ to $\mathcal{F}$ by: for each α, for each x in $[0, 1]$, $\psi^+_\alpha(x) = \lim_{n\to\infty} \psi^+_{\overline{\alpha}n}(x)$.

Also note that, for every α, for every p, n, $\psi^+_{\underline{0}n*\langle 2p+1\rangle*\alpha} = G_{\frac{1}{5},\frac{2}{5},\psi^+_{\underline{0}n*\langle 2p+1\rangle}}[\psi_\alpha]$ and $\psi^+_{\underline{0}n*\langle 2p+2\rangle*\alpha} = G_{\frac{3}{5},\frac{4}{5},\psi^+_{\underline{0}n*\langle 2p+2\rangle}}[\psi_\alpha]$.

Lemma 4. *For every subset $\mathcal{G}$ of $\mathcal{F}$ containing $\mathcal{I}_0$, if $\mathcal{G}$ has the invariance property and $\mathcal{G}$ is positively nonperhapsive, then $\mathcal{G}^+$ is positively nonperhapsive.*

Proof. Let $\mathcal{G}$ be a subset of $\mathcal{F}$ containing $\mathcal{I}_0$ that is positively nonperhapsive and enjoys the invariance property. Find a mapping $s \mapsto \psi_s$ from $\mathbb{N}^*$ to $\mathcal{F}$ such that, for every α in $\mathcal{N}$, for every x in $[0, 1]$, $\lim_{n\to\infty} \psi_{\overline{\alpha}n}(x)$ exists, and, if one defines a mapping $\alpha \mapsto \psi_\alpha$ from $\mathcal{N}$ to $\mathcal{F}$ by: for each α, for each x in $[0, 1]$, $\psi_\alpha(x) = \lim_{n\to\infty} \psi_{\overline{\alpha}n}(x)$, then, for every α, ψ_α belongs to $\mathcal{G}^+$ and not for every α, ψ_α belongs to $\mathcal{G}$.

Consider the mapping $s \mapsto \psi^+_s$ that we defined just before this theorem. We make two claims:

(i) For every α, ψ^+_α belongs to $\mathcal{G}^{++}$.

(ii) Not for every α, ψ^+_α belongs to $\mathcal{G}^+$.

Let us first prove the first claim. Assume that α belongs to $\mathcal{N}$ and that $\psi^+_\alpha(\frac{1}{2})\#\frac{1}{2}$. Find n, p such that α passes through either $\underline{0}n*\langle 2p+2\rangle$ or $\underline{0}n*\langle 2p+1\rangle$.

Suppose that α passes through $\underline{0}n*\langle 2p+2\rangle$. Find β such that $\alpha = \underline{0}n*\langle 2p+2\rangle*\beta$ and note that $\psi^+_\alpha = G_{\frac{3}{5},\frac{4}{5},\psi^+_{\underline{0}n*\langle 2p+2\rangle}}[\psi_\beta]$. As ψ_β belongs to $\mathcal{G}^+$ and $\mathcal{G}^+$ has, like $\mathcal{G}$ itself, the invariance property, see Lemma 2, also ψ^+_α belongs to $\mathcal{G}^+$.

Suppose that α passes through $\underline{0}n*\langle 2p+1\rangle$. Find β such that $\alpha = \underline{0}n*\langle 2p+1\rangle*\beta$ and note that $\psi^+_\alpha = G_{\frac{1}{5},\frac{2}{5},\psi^+_{\underline{0}n*\langle 2p+1\rangle}}[\psi_\beta]$. As ψ_β belongs to $\mathcal{G}^+$ and $\mathcal{G}^+$ has, like $\mathcal{G}$ itself, the invariance property, see Lemma 2, also ψ^+_α belongs to $\mathcal{G}^+$.

We thus see that, if $\psi^+_\alpha(\frac{1}{2})\#\frac{1}{2}$, then ψ^+_α belongs to $\mathcal{G}^+$. Therefore, ψ^+_α belongs to $\mathcal{G}^{++}$.

We now prove the second claim. Assume that, for all α, ψ^+_α belongs to $\mathcal{G}^+$. Then, for each α we may determine $i < 2$ such that, if $i = 0$, then there exists x in $[0, \frac{3}{5}]$ such that, if $\psi^+_\alpha(x)\#\frac{1}{2}$, then ψ^+_α belongs to $\mathcal{G}$, and, if $i = 1$, then there exists x in $[\frac{2}{5}, 1]$ such that, if $\psi^+_\alpha(x)\#\frac{1}{2}$, then ψ^+_α belongs to $\mathcal{G}$. Applying Brouwer's Continuity Principle, we find n, i such that, *either* $i = 0$ and, for every α, if $\overline{\alpha}n = \underline{0}n$, then there exists x in $[0, \frac{3}{5}]$ such that, if $\psi^+_\alpha(x)\#\frac{1}{2}$, then ψ^+_α

belongs to $\mathcal{G}$ *or* $i = 1$ and, for every α, if $\overline{\alpha}n = \overline{\underline{0}}n$, then there exists x in $[\frac{2}{5}, 1]$ such that, if $\psi^+_\alpha(x) \# \frac{1}{2}$, then ψ^+_α belongs to $\mathcal{G}$.

Let us first assume that we are in the first of these two cases, that is, $i = 0$. Observe that, for every β, for all x in $[0, \frac{3}{5}]$, $\psi^+_{\overline{\underline{0}}n*\langle 2p+2\rangle*\beta} < \frac{1}{2}$, and therefore, $\psi^+_{\overline{\underline{0}}n*\langle 2p+2\rangle*\beta}$ belongs to $\mathcal{G}$. Note that, for every β, $\psi^+_{\overline{\underline{0}}n*\langle 2p+2\rangle*\beta} = G_{\frac{3}{5},\frac{4}{5},\psi^+_{\overline{\underline{0}}n*\langle 2p+2\rangle}}[\psi_\beta]$. Using the fact that $\mathcal{G}$ has the invariance property, we find that, for every β, ψ_β belongs to $\mathcal{G}$ and thus a contradiction.

If we are in the second case, that is, $i = 1$, we also obtain a contradiction, in almost the same way. □

3.5 The first countably many perhapsive extensions of $\mathcal{I}_0$

Let us define a sequence $\mathcal{I}_0, \mathcal{I}_1, \mathcal{I}_2, \ldots$ of subclasses of $\mathcal{F}$ by: $\mathcal{I}_0$ is the class of all functions in $\mathcal{F}$ that assume the value $\frac{1}{2}$ and, for each n, $\mathcal{I}_{n+1} = (\mathcal{I}_n)^+$.

We call $\mathcal{I}_n$ the *n-th perhapsive extension of* $\mathcal{I}_0$.

Theorem 5. *(i) For each n, $\mathcal{I}_n \subseteq \mathcal{I}_{n+1} \subseteq (\mathcal{I}_0)^{\neg\neg}$.*
(ii) For each n, $\mathcal{I}_{n+1}$ is not a subset of $\mathcal{I}_n$.

Proof. (i) Use Lemma 1 and induction.
(ii) Use Lemmas 3 and 4 and induction. □

4 Forming limits

Lemma 6. *Let $\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2, \ldots$ be a sequence of subclasses of $\mathcal{F}$ such that, for each n, $\mathcal{I}_0 \subseteq \mathcal{A}_n \subseteq (\mathcal{I}_0)^{\neg\neg}$ and $\mathcal{A}_n$ is positively nonperhapsive and enjoys the invariance property, and, for each n, there exists p such that $(\mathcal{A}_n)^+ \subseteq \mathcal{A}_p$. Then also $\mathcal{I}_0 \subseteq \bigcup_{n\in\mathbb{N}} \mathcal{A}_n \subseteq (\mathcal{I}_0)^{\neg\neg}$ and $\bigcup_{n\in\mathbb{N}} \mathcal{A}_n$ is positively nonperhapsive and enjoys the invariance property.*

Proof. Let $s \mapsto \phi^0_s, s \mapsto \phi^1_s, s \mapsto \phi^0_s, \ldots$ be a sequence of mappings from $\mathbb{N}^*$ to $\mathcal{F}$ such that, for each n, for each α, for each x in $[0, 1]$, $\lim_{p\to\infty} \phi^n_{\overline{\alpha}p}(x)$ exists, and if one defines a mapping $\alpha \mapsto \psi^n_\alpha$ from $\mathcal{N}$ to $\mathcal{F}$ by: for each x in $[0, 1]$, $\phi^n_\alpha(x) = \lim_{p\to\infty} \phi^n_{\overline{\alpha}p}(x)$, then (i) for each α, ϕ^n_α belongs to $(\mathcal{A}_n)^+$ and (ii) not for every α, ϕ^n_α belongs to $\mathcal{A}_n$.

We now define another mapping from $\mathbb{N}^*$ to $\mathcal{F}$, giving it the name $s \mapsto \psi_s$, as follows:

(i) For all n, $\psi_{\overline{\underline{0}}n}$ is an element of $\mathcal{F}$ that is linear on each of the segments $[0, \frac{1}{3}]$, $[\frac{1}{3}, \frac{2}{3}]$ and $[\frac{2}{3}, 1]$ and takes the value $\frac{1}{2}$ at the points $\frac{1}{3}$ and $\frac{2}{3}$.

(ii) For all n, p in $\mathbb{N}$, for all s in $\mathbb{N}^*$, $\phi_{\underline{\overline{0}}n*\langle p+1\rangle*s}$ is an element of $\mathcal{F}$ that is linear on the segments $[0, \frac{1}{3}]$, and $[\frac{2}{3}, 1]$ and takes the value $\frac{1}{2} - \frac{1}{2^n}$ at the point $\frac{1}{3}$ and the value $\frac{1}{2} + \frac{1}{2^n}$ at the point $\frac{2}{3}$, and satisfies: for all x in $[\frac{1}{3}, \frac{2}{3}]$, $\phi_{\underline{\overline{0}}n*\langle p+1\rangle*s}(x) = \frac{1}{2} + \frac{1}{2^{n-1}}(\phi^p(3x-2) - \frac{1}{2})$.

Note that, for every α, $\lim_{n\to\infty} \psi_{\overline{\alpha}n}(x)$ exists. We thus may define a mapping $\alpha \mapsto \psi_\alpha$ from $\mathcal{N}$ to $\mathcal{F}$ by: for each α, for each x in $[0,1]$, $\psi_\alpha(x) = \lim_{n\to\infty} \psi_{\overline{\alpha}n}(x)$.

Also note that, for every α, for every p, n, $\psi_{\underline{\overline{0}}n*\langle p+1\rangle*\alpha} = G_{\frac{1}{3},\frac{2}{3},\psi_{\underline{\overline{0}}n*\langle p+1\rangle}}[\phi^p_\alpha]$.

We make two claims:

(i) For every α, ψ_α belongs to $(\bigcup_{n\in\mathbb{N}} \mathcal{A}_n)^+$.

(ii) Not for every α, ψ_α belongs to $\bigcup_{n\in\mathbb{N}} \mathcal{A}_n$.

Let us first prove the first claim. Assume that α belongs to $\mathcal{N}$ and that $\psi_\alpha(\frac{1}{2})\#\frac{1}{2}$. Find n, p such that α passes through $\underline{\overline{0}}n * \langle p+1\rangle$. Find β such that $\alpha = \underline{\overline{0}}n * \langle p+1\rangle * \beta$ and note that $\psi_\alpha = G_{\frac{1}{3},\frac{2}{3},\psi_{\underline{\overline{0}}n*\langle p+1\rangle}}[\phi^p_\beta]$. As ϕ^p_β belongs to $(\mathcal{A}_p)^+$ and $(\mathcal{A}_p)^+$ has, like $\mathcal{A}_p$ itself, the invariance property, see Lemma 2, also ψ_α belongs to $(\mathcal{A}_p)^+$. As, for some n, $(\mathcal{A}_p)^+$ is a subclass of $\mathcal{A}_n$, ψ_α also belongs to $\bigcup_{n\in\mathbb{N}} \mathcal{A}_n$. We thus see that, for every α, if $\psi_\alpha(\frac{1}{2})\#\frac{1}{2}$, then ψ_α belongs to $\bigcup_{n\in\mathbb{N}} \mathcal{A}_n$. It follows that, for every α, ψ_α belongs to $(\bigcup_{n\in\mathbb{N}} \mathcal{A}_n)^+$.

We now prove the second claim. Assume that, for every α, ψ_α belongs to $\bigcup_{n\in\mathbb{N}} \mathcal{A}_n$. Using Brouwer's Continuity Principle, we find p, n such that, for every α, if $\overline{\alpha}n = \underline{\overline{0}}n$, then ψ_α belongs to $\mathcal{A}_p$. It follows that, for every β, $\psi_{\underline{\overline{0}}n*\langle p\rangle*\beta} = G_{\frac{1}{3},\frac{2}{3},\psi_{\underline{\overline{0}}n*\langle p+1\rangle}}[\phi^p_\beta]$ belongs to $\mathcal{A}_p$. As $\mathcal{A}_p$ has the invariance property, it follows that, for every β, ϕ^p_β belongs to $\mathcal{A}_p$. Contradiction. □

We define: $\mathcal{I}_\omega = \bigcup_{n\in\mathbb{N}} \mathcal{I}_n$. We also define a sequence $\mathcal{I}_\omega, \mathcal{I}_{\omega+1}, \mathcal{I}_{\omega+2}, \ldots$ of subsets of $\mathcal{F}$ by: $\mathcal{I}_{\omega+n+1} = (\mathcal{I}_{\omega+n})^+$. (We are of course identifying $\mathcal{I}_{\omega+0}$ and $\mathcal{I}_\omega$.)

Theorem 7. *(i) $\mathcal{I}_\omega$ is positively non-perhapsive.*

(ii) For each n, $\mathcal{I}_{\omega+n}$ is positively nonperhapsive and a proper subclass of $\mathcal{I}_{\omega+n+1}$.

Proof. (i) is an easy consequence of Lemma 6.

(ii) now follows by Lemma 4. □

We may go further, of course, and, in the next Section, we discuss how to do so in a systematic way.

5 Labeling perhapsive extensions by means of stumps

Stumps are certain decidable subsets of the set $\mathbb{N}^*$. The definition of the set of stumps is inductive, as follows.

For every s in $\mathbb{N}^*$, for every subset A of $\mathbb{N}^*$, we let $s * A$ be the set of all finite sequences $s * t$, where t belongs to A.

We let $\langle\ \rangle$ denote the *empty sequence*, the only element of $\mathbb{N}^*$ of length 0.

(i) The empty set $\emptyset$ is a stump, sometimes called the *empty stump*.

(ii) For each infinite sequence $S_0, S_1, S_2, \ldots$ of stumps, the set $\{\langle\ \rangle\} \cup \bigcup_{n\in\mathbb{N}} \langle n\rangle * S_n$ is again a stump. The stumps $S_0, S_1, S_2, \ldots$ are called the *immediate substumps* of the stump $\{\langle\ \rangle\} \cup \bigcup_{n\in\mathbb{N}} \langle n\rangle * S_n$.

(iii) Every stump is obtained from the empty stump by the repeated application of the generating operation mentioned under (ii).

We use σ, τ, $\ldots$ as variables over the set of stumps.

For each non-empty stump σ, for each n, we let σ^n be the set of all t in $\mathbb{N}^*$ such that $\langle n\rangle * t$ belongs to σ. Note that σ^n is the n-th immediate substump of σ.

One may give proofs and define functions by induction on the set of stumps. We use the name **Stump** for the set of stumps.

First Principle of Induction and Recursion on the set of stumps:

(i) Let P be a subset of the set **Stump** *of stumps. If $\emptyset$ belongs to P, and each non-empty stump σ with the property that, for each n, σ^n belongs to P, belongs itself to P, then every stump belongs to P, that is, P coincides with* **Stump**.

(ii) Let G be a set and let $G^{\mathbb{N}}$ be the set of all infinite sequences of elements of G. Let g be an element of G and let $\mathbb{F}$ be an operation from $G^{\mathbb{N}}$ to G.

There exists an operation $\mathbb{H}$ from the set **Stump** *to the set G such that $\mathbb{H}(\emptyset) = g$ and, for each non-empty stump σ, $\mathbb{H}(\sigma) = \mathbb{F}(\lambda n \in \mathbb{N}.\mathbb{H}(\sigma^n))$.*

We define binary relations $<, \leq$ on the set **Stump** by simultaneous recursion:

(i) For every stump τ, $\emptyset \leq \tau$, and, for no stump σ, $\sigma < \emptyset$.

(ii) For every non-empty stump σ, for every stump τ, $\sigma \leq \tau$ if and only if, for all m, $\sigma^m < \tau$.

(iii) For every stump σ, for every non-empty stump τ, $\sigma < \tau$ if and only if, for some n, $\sigma \leq \tau^n$.

One now may prove that $<$ is a transitive relation on **Stump** satisfying the following

Second Principle of Induction on the set of stumps:

> *Let P be a subset of the set* **Stump** *of stumps. If each stump σ with the property that every stump $\tau < \sigma$ belongs to P belongs itself to P, then every stump belongs to P.*

Now we want to use stumps in order to label perhapsive extensions of $\mathcal{I}_0$.

First, we define, for every subset $\mathcal{G}$ of $\mathcal{F}$ containing $\mathcal{I}_0$, for every m, a subset $\mathcal{G}^{(m)}$ of $\mathcal{F}$, as follows, by induction: $\mathcal{G}^{(0)} = \mathcal{G}$, and, for each m, $\mathcal{G}^{(m+1)} = (\mathcal{G}^{(m)})^+$.

We define, for each stump σ, a subset $\mathbb{P}(\sigma, \mathcal{I}_0)$ of $\mathcal{F}$, calling it the *σ-th perhapsive extension of* $\mathcal{I}_0$, as follows, by induction:

(i) $\mathbb{P}(\emptyset, \mathcal{I}_0) = \mathcal{I}_0$

(ii) For each non-empty stump σ, $\mathbb{P}(\sigma, \mathcal{I}_0) = \bigcup_{m,n \in \mathbb{N}} \mathbb{P}(\sigma^n, \mathcal{I}_0)^{(m)}$

Theorem 8. *(i) For each stump σ, $\mathcal{I}_0 \subseteq \mathbb{P}(\sigma, \mathcal{I}_0) \subseteq (\mathcal{I}_0)^{\neg\neg}$ and $\mathbb{P}(\sigma, \mathcal{I}_0)$ is positively non-perhapsive.*

(ii) For every non-empty stump σ, for every n, $\mathbb{P}(\sigma^n, \mathcal{I}_0)$ is a proper subset of $\mathbb{P}(\sigma, \mathcal{I}_0)$.

(iii) For all stumps σ, τ, if $\sigma \leq \tau$, then $\mathbb{P}(\sigma, \mathcal{I}_0)$ is a subset of $\mathbb{P}(\tau, \mathcal{I}_0)$, and, if $\sigma < \tau$, then $\mathbb{P}(\sigma, \mathcal{I}_0)$ is a proper subset of $\mathbb{P}(\tau, \mathcal{I}_0)$.

Proof. The proofs are straightforward inductive arguments. One proves (i) using Lemmas 4 and 6. (ii) and (iii) then follow easily. □

6 Concluding remarks

6.1 It suffices to consider weakly monotone functions

Let $\mathcal{F}_{mon}$ be the class of all functions ϕ in $\mathcal{F}$ satisfying the condition of weak monotonicity: for all x, y in $[0, 1]$, if $x \leq y$ then $\phi(x) \leq \phi(y)$. Note that our arguments would go through if we should have restricted ourselves to $\mathcal{F}_{mon}$ rather than $\mathcal{F}$. In particular, if ψ, ρ both belong to $\mathcal{F}_{mon}$ and a, b are elements of $[0, 1]$ such that $a < b$, then also the function $G_{a,b,\rho}[\psi]$, as defined in Subsection 3.2, belongs to $\mathcal{F}_{mon}$.

6.2 The proper formal context

We have kept the style of this paper a bit informal. All results, however, may be formulated, for instance, in the language of the formal system BIM, (for *Basic Intuitionistic Mathematics*), introduced in [10]. We then would not take the notion of a function from $[0,1]$ to $\mathbb{R}$ as primitive. We would define a continuous function from $[0,1]$ to $\mathbb{R}$ as an enumeration of natural numbers, each of them coding a pair $\langle t, u\rangle$ of rational intervals. Of course, the enumeration has to satisfy certain natural conditions. The set $\mathcal{F}$ then would be introduced as a certain subset of Baire space $\mathcal{N}$.

6.3 On $(\mathcal{I}_0)^{\neg\neg}$

Does the set $\mathcal{F}$ coincide with the set $(\mathcal{I}_0)^{\neg\neg}$?

We have no satisfying answer to this question. If one assumes *Markov's Principle* in the form: "*for every* α*, if* $\neg\neg\exists n[\alpha(n) = 0]$*, then* $\exists n[\alpha(n) = 0]$", one may prove this statement. For suppose that ϕ belongs to $(\mathcal{I}_0)^{\neg}$. Using Markov's Principle one may assume that, for every x in $[0,1]$, $\phi(x)\#\frac{1}{2}$. Under this assumption one may construct, using the method of successive bisection, a point x such that $\phi(x) = \frac{1}{2}$. Contradiction. Clearly then, every ϕ in $\mathcal{F}$ belongs to $(\mathcal{I}_0)^{\neg\neg}$.

The intuitionistic mathematician does not see sufficient reason to defend Markov's Principle as an axiom for analysis. Some Russian constructivist do, however, see [1].

6.4 On $\bigcup_{\sigma\in\mathbf{Stump}} \mathbb{P}(\sigma, \mathcal{I}_0)$

Does the set $\mathcal{F}$ coincide with the set $\bigcup_{\sigma\in\mathbf{Stump}} \mathbb{P}(\sigma, \mathcal{I}_0)$?

Note that $\bigcup_{\sigma\in\mathbf{Stump}} \mathbb{P}(\sigma, \mathcal{I}_0)$ is a subset of $(\mathcal{I}_0)^{\neg\neg}$, so this question is related to the previous one. The answer to this question must be no. We give an outline of the argument. If these two sets coincide, and we have really constructive evidence that they do, we must be able to find a function g from $\mathcal{F}$ to **Stump** such that for every ϕ in $\mathcal{F}$, ϕ belongs to $\mathbb{P}(g(\phi), \mathcal{I}_0)$. It is important now that the set $\mathcal{F}$, as a subset of Baire space $\mathcal{N}$, is *strictly analytic*, that is, there exists a continuous mapping h from Baire space $\mathcal{N}$ onto $\mathcal{F}$. As a consequence, by an intuitionistic version of a famous *boundedness theorem* from descriptive set theory, see [8], there will exist a stump τ such that, for every ϕ in $\mathcal{F}$, $g(\phi) \leq \tau$, and $\mathcal{F}$ will be a subset of $\mathbb{P}(\tau, \mathcal{I}_0)$. This conclusion contradicts Theorem 8.

Combining this observation with the previous remark 6.3, we see that Markov's Principle implies that the sets $(\mathcal{I}_0)^{\neg\neg}$ and $\bigcup_{\sigma\in\mathbf{Stump}} \mathbb{P}(\sigma, \mathcal{I}_0)$ do not coincide.

6.5 Almost$(\mathcal{I}_0)$

We now introduce the name Almost$(\mathcal{I}_0)$ for the set $\bigcup_{\sigma\in\mathbf{Stump}} \mathbb{P}(\sigma, \mathcal{I}_0)$. This terminology agrees with the terminology used in [8]. We want to sketch a characterization of this set.

First, one may prove that, for every ϕ in $\mathcal{F}$, ϕ belongs to $\mathcal{I}_1 = (\mathcal{I}_0)^+$ if and and only if there exists a sequence $x_0, x_1, x_2, \ldots$ of elements of $[0,1]$ with the properties:

(i) for each n, either $x_n = x_{n+1}$ or $x_n \# x_{n+1}$,

(ii) for each n, $\phi(x_n)\#\frac{1}{2}$ if and only if, for some $m \geq n$, $x_m \# x_{m+1}$, and

(iii) there is at most one number n such that $x_n \# x_{n+1}$.

(The classical mathematician would say: "*the sequence* $x_0, x_1, x_2, \ldots$ *is convergent and* ϕ *assumes the value* $\frac{1}{2}$ *at the point* $x = \lim_{n\to\infty} x_n$". From a constructive point of view however, a sequence with the property (iii) is not necessarily convergent.)

Next, one may prove that, for every k, for every ϕ in $\mathcal{F}$, ϕ belongs to $\mathcal{I}_k = (\mathcal{I}_0)^{(k)}$ if and and only if there exists a sequence $x_0, x_1, x_2, \ldots$ of elements of $[0,1]$ with the properties:

(i) for each n, either $x_n = x_{n+1}$ or $x_n \# x_{n+1}$,

(ii) for each n, $\phi(x_n)\#\frac{1}{2}$ if and only if, for some $m \geq n$, $x_m \# x_{m+1}$, and

(iii) there are at most k numbers n such that $x_n \# x_{n+1}$.

In [5], [6], [8] and [9] study is made of decidable subsets of $\mathbb{N}$ that are *almost-finite*. It follows from Brouwer's Thesis on bars that a decidable subset A of $\mathbb{N}$ is almost-finite if and only if, for every strictly increasing γ in $\mathcal{N}$, there exists n such that $\gamma(n)$ does not belong to A. One could say that A is almost-finite if and only if we are sure that every attempt to prove that A has an infinite subset will fail after finitely many steps.

One now may show that, for every ϕ in $\mathcal{F}$, ϕ belongs to Almost$(\mathcal{I}_0)$ if and and only if there exists a sequence $x_0, x_1, x_2, \ldots$ of elements of $[0,1]$ with the properties:

(i) for each n, either $x_n = x_{n+1}$ or $x_n \# x_{n+1}$,

(ii) for each n, $\phi(x_n)\#\frac{1}{2}$ if and only if, for some $m \geq n$, $x_m \# x_{m+1}$, and

(iii) the set of all numbers n such that $x_n \# x_{n+1}$ is an almost-finite subset of $\mathbb{N}$.

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