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Completeness in the Boolean Hierarchy: Exact-Four-Colorability, Minimal Graph Uncolorability, and Exact Domatic Number Problems – a Survey

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Abstract: This paper surveys some of the work that was inspired by Wagner's general technique to prove completeness in the levels of the boolean hierarchy over NP and some related results. In particular, we show that it is DP-complete to decide whether or not a given graph can be colored with exactly four colors, where DP is the second level of the boolean hierarchy. This result solves a question raised by Wagner in 1987, and its proof uses a clever reduction due to Guruswami and Khanna. Another result covered is due to Cai and Meyer: The graph minimal uncolorability problem is also DP-complete. Finally, similar results on various versions of the exact domatic number problem are discussed.
Key Words: Boolean hierarchy, completeness, exact colorability, exact domatic number, minimal uncolorability.
Category: F.1.2, F.1.3, F.2.2, F.2.3

1 Introduction, Historical Notes, and Definitions

This paper surveys completeness results in the levels of the boolean hierarchy over NP, with a special focus on Wagner's work [Wag87]. His general technique for proving completeness in the boolean hierarchy levels—as well as in other classes such as $\mathrm{P}_{||}^{\mathrm{NP}}$, the class of problems solvable via parallel access to NP—inspired much of the recent results in this area. Quoting Papadimitriou, the boolean hierarchy is "somewhat sparse in natural complete sets" (see p. 434 of [Pap94]). This statement certainly is true—in particular, if the number of natural problems complete in higher boolean hierarchy levels is set off against the number of natural NP-complete problems. However, even the higher levels of the boolean hierarchy do contain very natural, beautiful complete problems, and this survey's goal is to present some of them. Of course, as there are only few of them known, we should seek to find more. This line of research has been intensely pursued since the late 1980s, and much work has been done in a number of recent papers. The purpose of the present survey is to give an overview of this progress of results.

But first, let us look back a bit further and start with the beginning. In the 1970s, Meyer and Stockmeyer [MS72] studied the problem `Minimal`, which for a given boolean formula φ asks whether there is no shorter formula equivalent to φ. They noted that this problem can be accepted by a coNP machine accessing an NP oracle, thus creating the second level of the polynomial hierarchy, which consists of the classes $\Sigma_2^p = \mathrm{NP}^{\mathrm{NP}}$ and $\Pi_2^p = \mathrm{coNP}^{\mathrm{NP}}$. Motivated by this observation, they introduced the polynomial hierarchy in order to capture the complexity of problems that appear to be beyond NP and coNP. Figure 1 shows the inclusion structure of the polynomial hierarchy.

Definition 1 (Polynomial Hierarchy). The *polynomial hierarchy* is inductively defined by:

- $\Delta_0^p = \Sigma_0^p = \Pi_0^p = \mathrm{P}$,
- for $i \geq 0$, $\Delta_{i+1}^p = \mathrm{P}^{\Sigma_i^p}$, $\Sigma_{i+1}^p = \mathrm{NP}^{\Sigma_i^p}$, and $\Pi_{i+1}^p = \mathrm{co}\Sigma_{i+1}^p$, and
- $\mathrm{PH} = \bigcup_{k \geq 0} \Sigma_k^p$.

Variants of the problem `Minimal` have been studied as well. Garey and Johnson [GJ79] defined the minimum equivalent expression problem (`MEE`, for short): Given a boolean formula φ and a nonnegative integer k, does there exist a boolean formula ψ with at most k literals such that ψ is equivalent to φ? Stockmeyer [Sto77] considered the restriction of `MEE` to boolean formulas in disjunctive normal form (DNF), which we here denote by `MEE-DNF`. It is not hard to see that both `MEE` and `MEE-DNF` are contained in Σ_2^p, but the question of whether `MEE-DNF` is Σ_2^p-complete was open for more than two decades, and for `MEE` this question is still open today. The best known lower bounds (i.e., hardness results) for the three problems just defined are stated in Section 2.

In this paper, all hardness and completeness results are with respect to the polynomial-time many-one reducibility, denoted by $\leq_{\mathrm{m}}^{\mathrm{p}}$: For sets A and B, we write $A \leq_{\mathrm{m}}^{\mathrm{p}} B$ if and only if there is a polynomial-time computable function f such that for each $x \in \Sigma^*$, $x \in A$ if and only if $f(x) \in B$. A set B is said to be $\mathcal{C}$*-hard* for a complexity class $\mathcal{C}$ if and only if $A \leq_{\mathrm{m}}^{\mathrm{p}} B$ for each $A \in \mathcal{C}$. A set B is said to be $\mathcal{C}$*-complete* if and only if B is $\mathcal{C}$-hard and $B \in \mathcal{C}$.

Papadimitriou and Zachos [PZ83] introduced $\mathrm{P}^{\mathrm{NP}[\mathcal{O}(\log)]}$, the class of problems solvable by $\mathcal{O}(\log n)$ sequential Turing queries to NP. Köbler, Schöning, and Wagner [KSW87] and, independently, Hemaspaandra [Hem87] proved that $\mathrm{P}^{\mathrm{NP}[\mathcal{O}(\log)]}$ equals $\mathrm{P}_{||}^{\mathrm{NP}}$, the class of problems solvable by parallel (a.k.a. truth-table) access to NP. Wagner [Wag90] provided about half a dozen other characterizations of this class, and he introduced the notation Θ_2^p for it. By definition, $\mathrm{NP} \subseteq \Theta_2^p \subseteq \Delta_2^p$. It is known that if NP contains some problem that is hard for Θ_2^p, then the polynomial hierarchy collapses to NP, see Meyer and Stockmeyer [MS72, Sto77]. The class Θ_2^p is also closely related to the question of whether NP has sparse Turing-hard sets [Kad89], and to various other topics; see, e.g., [LS95, Kre88, HW91]. Wagner also introduced the classes $\Theta_i^p = \mathrm{P}^{\Sigma_{i-1}^p[\mathcal{O}(\log)]}$

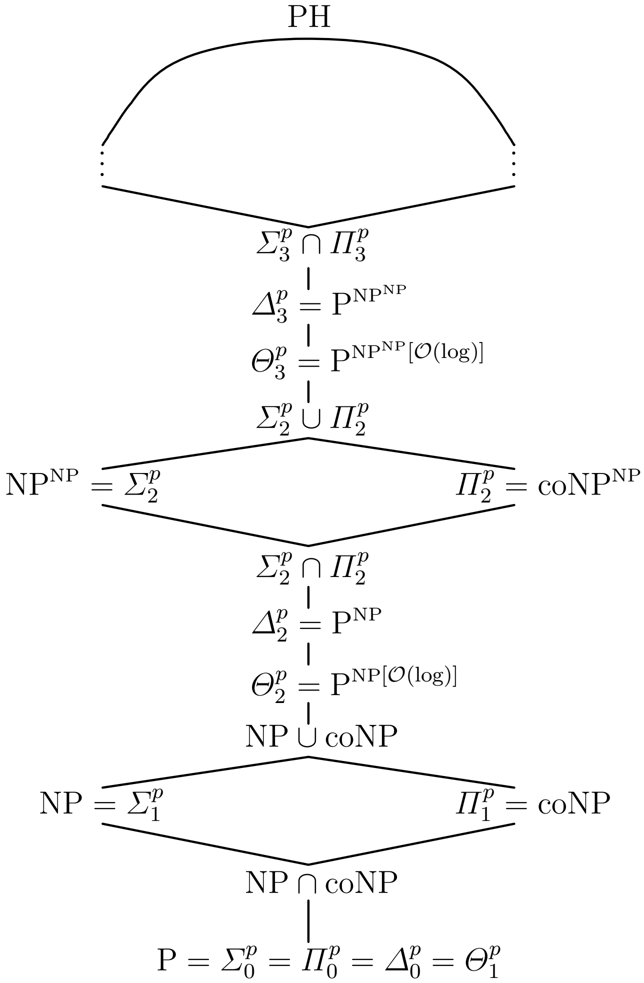

Figure 1: The polynomial hierarchy

for each $i \geq 1$, as a straightforward generalization of Θ_2^p to higher levels of the polynomial hierarchy.

In the 1980s, Papadimitriou and Yannakakis [PY84] noted that certain NP-hard and coNP-hard problems seem to be not complete for NP or coNP:

- *Exact problems* such as `Exact-4-Colorability`: Given a graph, is it true that it can be legally colored with *exactly* four colors? (See Definition 3 below.)
- *Critical Problems* such as `Minimal-3-Uncolorability`: Given a graph, is it true that it is not 3-colorable, yet deleting any of its vertices makes it 3-colorable? (See Definition 10 in Section 4.)

– *Unique solution problems* such as `Unique-SAT`: Given a boolean formula, is it true that it has exactly one satisfying assignment?

Motivated by this observation, they introduced the class of differences of NP sets:

$$\mathrm{DP} = \{A - B \mid A, B \in \mathrm{NP}\}.$$

All the above problems are in DP.

The complexity of colorability problems has been studied intensely, see, e.g., [AH77a, AH77b, Sto73, GJS76, Wag87, KV91, Rot00, GRW01a, GRW01b, Rot03].

Definition 2 (Colorability Problem). For any graph G with vertex set $V(G)$ and edge set $E(G)$, a *k-coloring of G* is a partition $V(G) = V_1 \cup V_2 \cup \cdots \cup V_k$ of the vertex set $V(G)$ into k disjoint sets. A k-coloring is called *legal* if for $1 \leq i \leq k$, every set V_i is an independent set, i.e., there is no edge in $E(G)$ between any pair of vertices in V_i. Define $\chi(G)$ to be the *chromatic number of G*, i.e., the smallest number of colors needed to legally color G. For each k, we further define

$$k\text{-}\mathtt{Colorability} = \{G \mid G \text{ is a graph with } \chi(G) \leq k\}.$$

The problem `2-Colorability` is in P, yet `3-Colorability` is NP-complete, see Stockmeyer [Sto73]. We now define the exact versions of colorability problems.

Definition 3 (Exact Colorability Problems). Let M_k be a set that consists of k integers, and let t be a positive integer. Define

$$\mathtt{Exact}\text{-}M_k\text{-}\mathtt{Colorability} = \{G \mid G \text{ is a graph with } \chi(G) \in M_k\},$$
$$\mathtt{Exact}\text{-}t\text{-}\mathtt{Colorability} = \{G \mid G \text{ is a graph with } \chi(G) = t\}.$$

Merging, unifying, and expanding the results that originally were obtained independently by Cai and Hemaspaandra [CH86] and by Gundermann, Wagner, and Wechsung [Wec85, GW87], Cai et al. [CGH$^+$88, CGH$^+$89] generalized DP by introducing the boolean hierarchy over NP.[1] To define this hierarchy, we use the symbols $\wedge$ and $\vee$, respectively, to denote the *complex intersection* and the *complex union* of set classes:

$$\mathcal{C} \wedge \mathcal{D} = \{A \cap B \mid A \in \mathcal{C} \text{ and } B \in \mathcal{D}\};$$
$$\mathcal{C} \vee \mathcal{D} = \{A \cup B \mid A \in \mathcal{C} \text{ and } B \in \mathcal{D}\}.$$

[1] As a historical note, Cai and Hemaspaandra [CH86] introduced the boolean hierarchy as "hardware over NP." Gundermann, Wagner, and Wechsung independently studied this hierarchy, motivated mainly by "counting classes with finite acceptance types," see [Wec85, GW87] (and also [GNW90] for a follow-up paper along these lines of research). Out of these early papers grew a close collaboration between the two groups of researchers and the joint work by Cai et al. [CGH$^+$88, CGH$^+$89], which provides the perhaps most comprehensive list of results on the boolean hierarchy.

Definition 4 (Boolean Hierarchy over NP). The *boolean hierarchy over* NP is inductively defined by:

$$\mathrm{BH}_0(\mathrm{NP}) = \mathrm{P}, \quad \mathrm{BH}_1(\mathrm{NP}) = \mathrm{NP}, \quad \mathrm{BH}_2(\mathrm{NP}) = \mathrm{NP} \wedge \mathrm{coNP} = \mathrm{DP},$$
$$\mathrm{BH}_k(\mathrm{NP}) = \mathrm{BH}_{k-2}(\mathrm{NP}) \vee \mathrm{BH}_2(\mathrm{NP}) \quad \text{for } k \geq 3, \text{ and}$$
$$\mathrm{BH}(\mathrm{NP}) = \bigcup_{k \geq 1} \mathrm{BH}_k(\mathrm{NP}).$$

Figure 2 shows the inclusion structure of the boolean hierarchy. Note further that $\mathrm{BH}(\mathrm{NP}) \subseteq \Theta_2^p \subseteq \Delta_2^p \subseteq \Sigma_2^p \subseteq \mathrm{PH}$. Kadin [Kad88] was the first to show that a collapse of the boolean hierarchy implies a collapse of the polynomial hierarchy. Both figures—Figure 1 for the polynomial hierarchy and Figure 2 for the boolean hierarchy—are extended versions of figures from [Rot05].

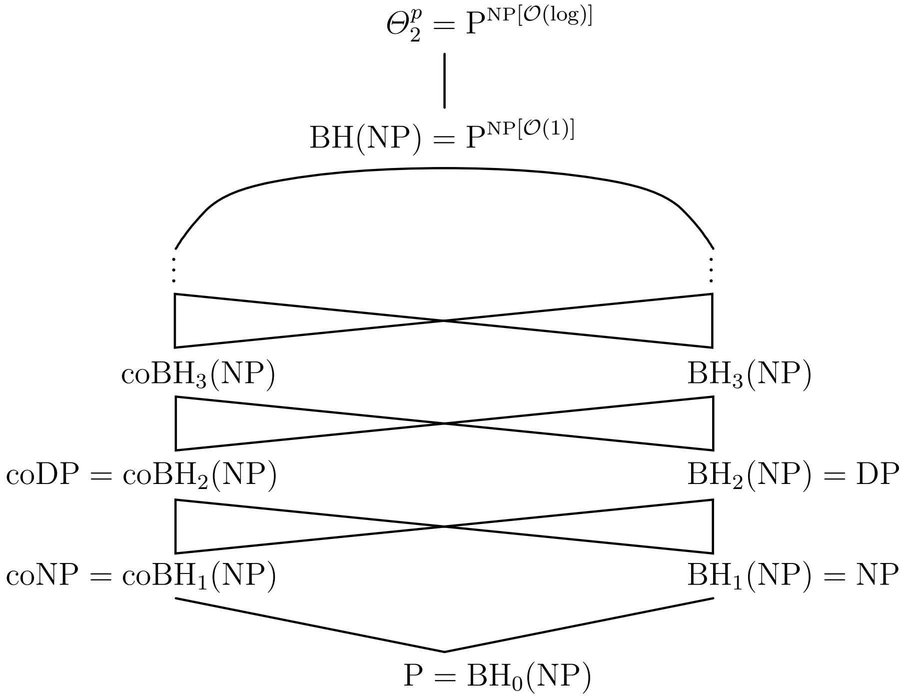

Figure 2: The boolean hierarchy over NP

Theorem 5 (Kadin [Kad88]). *If* $\mathrm{BH}_k(\mathrm{NP}) = \mathrm{coBH}_k(\mathrm{NP})$ *for some* $k \geq 1$, *then the polynomial hierarchy collapses to its third level:* $\mathrm{PH} = \Sigma_3^p \cap \Pi_3^p$.

The collapse consequence of Theorem 5 has been strenghtened later on; see the survey by Hemaspaandra, Hemaspaandra, and Hempel [HHH98].

2 Some Results Obtained by Wagner's Technique

Wagner [Wag87] established conditions sufficient to prove hardness for Θ_2^p and for the levels of the boolean hierarchy over NP. We first state his sufficient condition for proving Θ_2^p-hardness.

Lemma 6 (Wagner [Wag87]). *Let A be some* NP*-complete set, and let B be any set. If there exists a polynomial-time computable function g such that for all $\varphi_1, \ldots, \varphi_k$ in Σ^* with $(\forall j : 1 \leq j < k)\, [\varphi_{j+1} \in A \implies \varphi_j \in A]$ it holds that*

$$||\{i \mid \varphi_i \in A\}|| \text{ is odd } \iff g(\varphi_1, \ldots, \varphi_k) \in B, \tag{2.1}$$

then B is Θ_2^p-hard.

Using Lemma 6, Wagner proved dozens of problems Θ_2^p-complete, including the following variants of the colorability problem:

$$\begin{aligned}
\texttt{Color}_{\texttt{odd}} &= \{G \mid G \text{ is a graph such that } \chi(G) \text{ is odd}\},\\
\texttt{Color}_{\texttt{equ}} &= \{\langle G, H\rangle \mid G \text{ and } H \text{ are graphs with } \chi(G) = \chi(H)\},\\
\texttt{Color}_{\texttt{leq}} &= \{\langle G, H\rangle \mid G \text{ and } H \text{ are graphs with } \chi(G) \leq \chi(H)\}.
\end{aligned}$$

Wagner's technique has been applied to prove further natural problems, which arise in a variety of contexts, Θ_2^p-hard or even Θ_2^p-complete.[2] For example, Lemma 6 was useful in determining the complexity of the winner problem for certain voting systems, including Carroll elections [HHR97a], Young elections [RSV03], and Kemeny elections [HSV05]. For more background on computational politics, see Hemaspaandra and Hemaspaandra's excellent survey [HH00] and, e.g., [BTT89a, BTT89b, BTT92, CS02a, CS02b, CLS03, HHR05].

Wagner's technique was also useful for showing that recognizing those graphs for which certain efficient approximation heuristics for the independent set and the vertex cover problem do well is Θ_2^p-complete [HR98, HRS06]; see also the survey [HHR97b]. Moreover, Lemma 6 is the key lemma for raising the trivial hardness results for some of the three minimum equivalent expression problems defined in the introduction. In particular, Hemaspaandra and Wechsung [HW97, HW02] proved that `MEE` and `MEE-DNF` both are Θ_2^p-hard, and they also showed that `Minimal` is coNP-hard. Using a different technique, Umans [Uma01] proved that `MEE-DNF` is even Σ_2^p-complete. The precise complexity of `MEE` is still unknown today.

In what follows, we focus on completeness for exact colorability, minimal uncolorability, and exact domatic number problems in the even levels of the boolean hierarchy. The following lemma, which is also due to Wagner [Wag87], is the key lemma to establish most of these results.

[2] For other approaches to Θ_2^p-completeness, see, e.g., Krentel [Kre88], Eiter and Gottlob [EG97], and Spakowski and Vogel [SV00].

Lemma 7 (Wagner [Wag87]). *Let A be some* NP*-complete set, let B be any set, and let $k \geq 1$ be fixed. If there exists a polynomial-time computable function g such that for all $\varphi_1, \ldots, \varphi_{2k}$ in Σ^* with $(\forall j : 1 \leq j < 2k)\,[\varphi_{j+1} \in A \implies \varphi_j \in A]$ it holds that*

$$||\{i \mid \varphi_i \in A\}|| \text{ is odd } \iff g(\varphi_1, \ldots, \varphi_{2k}) \in B, \tag{2.2}$$

then B is $\mathrm{BH}_{2k}(\mathrm{NP})$*-hard.*

3 Exact Colorability Problems

In this section, we turn to the exact colorability problems defined in Definition 3. Using Lemma 7, Wagner [Wag87] proved the following result.

Theorem 8 (Wagner [Wag87]). *For $M_k = \{6k+1, 6k+3, \ldots, 8k-1\}$, the problem* `Exact-`$M_k$`-Colorability` *is* $\mathrm{BH}_{2k}(\mathrm{NP})$*-complete. In particular, for $k = 1$, it is* DP*-complete to determine whether or not $\chi(G) = 7$.*

Wagner [Wag87] raised the following questions: How small can the numbers in a k-element set M_k be chosen so as to ensure that `Exact-`M_k`-Colorability` still is $\mathrm{BH}_{2k}(\mathrm{NP})$-complete? In particular, for $k = 1$, is there some threshold $t \in \{4, 5, 6, 7\}$ such that `Exact-`t`-Colorability` jumps from NP to DP-complete? For example, is it DP-complete to determine whether or not $\chi(G) = 4$? Or is the complexity of `Exact-`t`-Colorability`, $4 \leq t \leq 6$, "intermediate" between NP and DP-complete?

These questions have been answered recently, see Rothe [Rot03]. Note that `Exact-3-Colorability` is in NP and thus cannot be DP-complete, unless the boolean hierarchy over NP (and, by Theorem 5, the polynomial hierarchy as well) collapses.

Theorem 9 (Rothe [Rot03]). *For $M_k = \{3k+1, 3k+3, \ldots, 5k-1\}$, the problem* `Exact-`$M_k$`-Colorability` *is* $\mathrm{BH}_{2k}(\mathrm{NP})$*-complete. In particular, for $k = 1$, it is* DP*-complete to determine whether or not $\chi(G) = 4$.*

A proof sketch for Theorem 9 is presented in the remainder of this section. Figures 4 through 9 have been taken (in slightly modified form) from [GK00] to illustrate the proof sketch. Crucially, this proof uses:

- Wagner's tool for proving $\mathrm{BH}_{2k}(\mathrm{NP})$-hardness stated as Lemma 7 above,
- the standard reduction f from `3-SAT` to `3-Colorability` satisfying

$$\varphi \in \texttt{3-SAT} \implies \chi(f(\varphi)) = 3, \tag{3.3}$$

$$\varphi \notin \texttt{3-SAT} \implies \chi(f(\varphi)) = 4, \tag{3.4}$$

– and Guruswami and Khanna's reduction g from 3-SAT to 3-Colorability satisfying

$$\varphi \in \texttt{3-SAT} \implies \chi(g(\varphi)) = 3, \tag{3.5}$$
$$\varphi \notin \texttt{3-SAT} \implies \chi(g(\varphi)) = 5. \tag{3.6}$$

Among the above three items, the Guruswami–Khanna reduction is the technically most challenging one. Originally, Guruswami and Khanna's seminal result is not motivated by the issue of proving the hardness of exact colorability. Rather, it was motivated by issues related to the hardness of approximating the chromatic number of 3-colorable graphs. Intuitively, their result says that it is NP-hard to 4-color a 3-colorable graph. This result had been obtained earlier on by Khanna, Linial, and Safra [KLS00] using the PCP theorem, which is due to Arora, Lund, Motwani, Sudan, and Szegedy [ALM$^+$98]. Guruswami and Khanna [GK00] gave a novel proof of this result, which does not rely on the PCP theorem. Their direct transformation in fact consists of the following two subsequent reductions:

$$\texttt{3-SAT} \leq_{\mathrm{m}}^{\mathrm{p}} \texttt{IS} \leq_{\mathrm{m}}^{\mathrm{p}} \texttt{3-Colorability},$$

where IS is the independent set problem: Given a graph G and a positive integer k, does G have an independent set of size at least k?

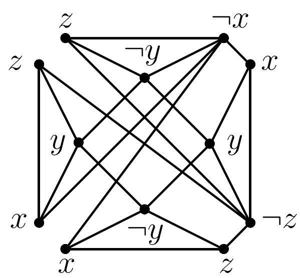

Figure 3: Graph G in the reduction $\texttt{3-SAT} \leq_{\mathrm{m}}^{\mathrm{p}} \texttt{IS}$

Figure 3 shows the standard reduction $\texttt{3-SAT} \leq_{\mathrm{m}}^{\mathrm{p}} \texttt{IS}$, for the specific formula

$$\varphi(x, y, z) = (x \vee y \vee z) \wedge (\neg x \vee \neg y \vee z) \wedge (x \vee y \vee \neg z) \wedge (x \vee \neg y \vee z).$$

Clauses in the formula correspond to triangles in the graph constructed, and corners of two distinct triangles are connected by an edge if and only if they correspond to some literal and its negation. Suppose the given formula has m

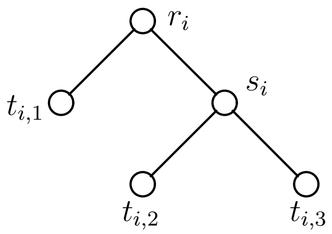

Figure 4: Tree-like structure S_i in the Guruswami–Khanna reduction

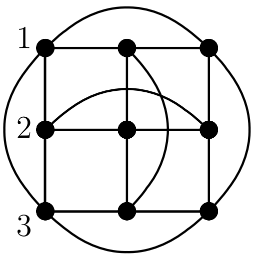

Figure 5: Basic template in the Guruswami–Khanna reduction

clauses, and denote the corresponding m triangles in G by $T_1, T_2, \ldots, T_m$. To each T_i in G, there corresponds a tree-like structure S_i as shown in Figure 4:

The three "leaves" $t_{i,1}$, $t_{i,2}$, and $t_{i,3}$ in S_i correspond to the three corners of the triangle T_i. Every "vertex" of S_i has the form of the basic template, which is a 3×3 grid such that the vertices in each row and column induce a 3-clique as shown in Figure 5: The "ground vertices" in the first column of any such basic template in fact are shared among all basic templates in each of the tree-like structures. Since these ground vertices form a 3-clique, every legal coloring assigns three distinct colors to them, say 1, 2, and 3.

Figure 6 shows the connection pattern between the "vertices" r_i, $t_{i,1}$, and s_i of S_i and two additional triangles. An analogous pattern applies to s_i, $t_{i,2}$, and $t_{i,3}$. Every vertex of the templates and the triangles is labeled by a triple of colors, and the vertices are connected according to the following simple rule: Two vertices are adjacent if and only if their labels differ in each coordinate.

A "vertex" in some S_i is said to be *selected (with respect to some coloring)* if and only if at least one of the three rows in its basic template receives colors that form an even permutation of $\{1, 2, 3\}$. That is, a "vertex" is selected if and only if

- the first row has colors $1, 2, 3$ from left to right, or
- the second row has colors $2, 3, 1$ from left to right, or

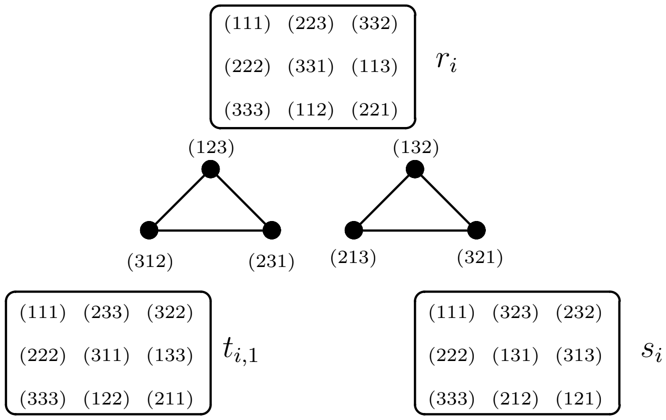

Figure 6: Connection pattern between the templates of a tree-like structure

– the third row has colors $3, 1, 2$ from left to right.

Note that for each legal 4-coloring of S_i, every "vertex" is either selected or not selected. Adding three more edges to each "vertex" r_i, the selection of every 3×3 root grid is enforced, as is shown in Figure 7. From the way the grids are connected, it follows that for any legal 4-coloring, selection of an internal "vertex" is propagated to at least one of its children. Therefore at least one of the "leaves" $t_{i,1}$, $t_{i,2}$, and $t_{i,3}$ must be selected as well. Additionally, it can be shown that for each "leaf" $t_{i,j}$, $1 \leq j \leq 3$, in a tree-structure S_i, there exists a legal 3-coloring of the vertices of S_i, where $t_{i,j}$ is the only "leaf" selected; see Properties (a) and (b) stated below.

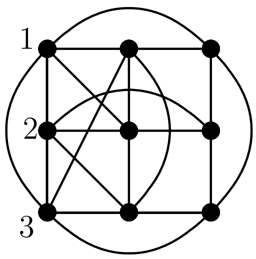

Figure 7: The root grid altered such that selection is enforced

The intuition of how to connect S_i and S_j, for distinct i and j, is as follows. For each pair of "vertices," $t_{i,k}$ and $t_{j,\ell}$, that are adjacent in graph G, appropriate gadgets are inserted to prevent that both these "leaves" are selected simultaneously, for otherwise G would have an independent set of size m if the graph constructed were 4-colorable.

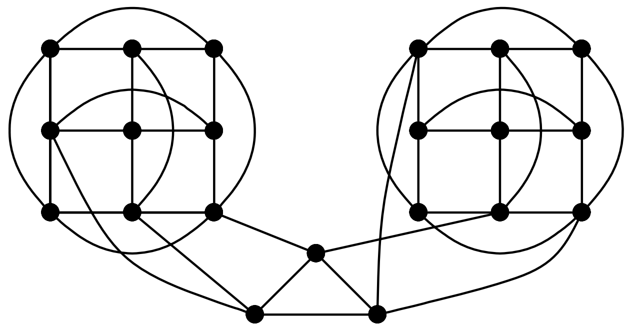

Figure 8: Gadget connecting two "leaves" of the same row kind

To this end, two kinds of gadgets are used, the *"same row" gadget* and the *"different rows" gadget*. Figure 8 shows the same row gadget, which prevents that $t_{i,k}$ and $t_{j,\ell}$ are simultaneously selected because of the same row. Figure 9 shows the different rows gadget, which prevents that $t_{i,k}$ and $t_{j,\ell}$ are selected simultaneously because of different rows.

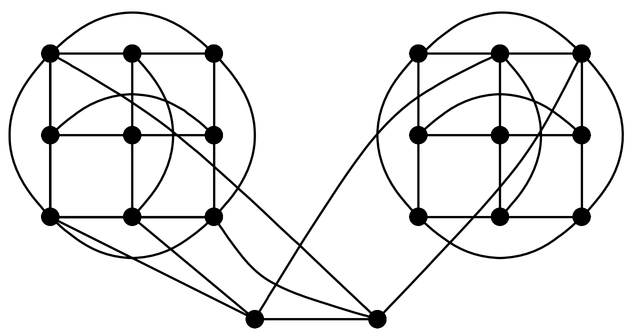

Figure 9: Gadget connecting two "leaves" of the different rows kind

This completes the reduction g that transforms the formula φ via graph G to graph $H = g(\varphi)$. We omit the detailed argument of why this reduction works to prove (3.5) and (3.6), referring to Guruswami and Khanna [GK00] instead. We merely mention that it can be shown that:

(a) For each i with $1 \leq i \leq m$, there exists a legal 3-coloring of the vertices in

S_i selecting exactly one of the three "leaves," $t_{i,1}$, $t_{i,2}$, and $t_{i,3}$.

(b) Every legal 4-coloring of S_i selects at least one of $t_{i,1}$, $t_{i,2}$, or $t_{i,3}$.

The implications (3.5) and (3.6) follow from (a) and (b).

Note that Guruswami and Khanna claimed in their conference paper [GK00] that $\varphi \notin$ 3-SAT implies $5 \leq \chi(H) \leq 6$. However, as has been observed in [Rot03], the Guruswami–Khanna reduction even yields the stronger implication (3.6), which is needed in order to apply Wagner's Lemma 7.

We are now ready to apply Lemma 7 with $k = 1$, and the sets $A =$ 3-SAT and $B =$ Exact-4-Colorability. Given two formulas φ_1 and φ_2 satisfying

$$\varphi_2 \in \texttt{3-SAT} \implies \varphi_1 \in \texttt{3-SAT}, \tag{3.7}$$

define the graphs $H_1 = g(\varphi_1)$ and $H_2 = f(\varphi_2)$, where g is the Guruswami–Khanna reduction, which satisfies (3.5) and (3.6), and f is the standard reduction from 3-SAT to 3-Colorability, which satisfies (3.3) and (3.4).

Let D be the disjoint union of H_1 and H_2. Thus,

$$\chi(D) = \max\{\chi(H_1), \chi(H_2)\}.$$

Consider the following three cases:

– If $\varphi_1 \in$ 3-SAT and $\varphi_2 \in$ 3-SAT, then $\chi(\varphi_1) = 3$ and $\chi(\varphi_2) = 3$, so $\chi(D) = 3$.

– If $\varphi_1 \in$ 3-SAT and $\varphi_2 \notin$ 3-SAT, then $\chi(\varphi_1) = 3$ and $\chi(\varphi_2) = 4$, so $\chi(D) = 4$.

– If $\varphi_1 \notin$ 3-SAT and $\varphi_2 \notin$ 3-SAT, then $\chi(\varphi_1) = 5$ and $\chi(\varphi_2) = 4$, so $\chi(D) = 5$.

By (3.7), the case distinction is complete. It follows that (2.2) is satisfied. By Lemma 7, Exact-4-Colorability is DP-hard. Since Exact-4-Colorability is in DP, it is DP-complete. Completeness of Exact-M_k-Colorability in $\mathrm{BH}_{2k}(\mathrm{NP})$ for the k-element set $M_k = \{3k+1, 3k+3, \ldots, 5k-1\}$ is proven analogously.

4 The Graph Minimal Uncolorability Problem

This section presents a well-known and typical example of a critical graph problem. A graph G is said to be *critical* if and only if by deleting any one of the vertices of G (respectively, by adding one vertex to G), the graph gains a certain property that it did not have before the removal (respectively, before the insertion) of this vertex. Similarly, one can define critical graph problems with respect to adding or removing edges in such a way that a specific property of the graph is triggered. Critical problems[3] are good candidates for DP-completeness; usually, these problems are easily shown to be contained in DP. Our first example of a critical problem is given below.

[3] The class of critical problems is not restricted to graph problems but can be defined in a broader sense. Here, however, we focus on some particularly interesting critical graph problem.

Definition 10 (Graph Minimal Uncolorability). Define the critical graph problem `Minimal-`k`-Uncolorability` as follows: Given a graph G, is it true that $G \notin k$-`Colorability`, but for every vertex $v \in V(G)$ it holds that $G - \{v\}$ is in k-`Colorability`? Here, $G - \{v\}$ denotes the induced subgraph that is obtained from G by deleting v from $V(G)$ and all incident edges from $E(G)$.

We are interested in the particular problem `Minimal-3-Uncolorability`, and we use `M-3-UC` as a shorthand for this problem. The following theorem is due to Cai and Meyer [CM87]. To prove DP-hardness of `M-3-UC`, they give a reduction from the problem `Minimal-3-UNSAT`, which was shown to be DP-complete by Papadimitriou and Wolfe [PW88]. The `Minimal-3-UNSAT` problem asks, given a boolean formula φ whose clauses contain exactly three literals each, is it true that φ is not satisfiable, but removing any one of its clauses makes φ satisfiable?

Theorem 11 (Cai and Meyer [CM87]). *The problem* `M-3-UC` *is* DP*-complete.*

To see that `M-3-UC` is in DP, consider the two sets

$$A = \{G \mid G \text{ is a graph with } \chi(G - \{v\}) \leq 3 \text{ for all vertices } v \in V(G)\} \quad \text{and}$$
$$B = \{G \mid G \text{ is a graph with } \chi(G) > 3\}.$$

Note that A is in NP, and B (which is the complement of an NP set) is in coNP, It is `M-3-UC` $= A \cap B$. The remainder of this section sketches Cai and Meyer's reduction from `Minimal-3-UNSAT` to `M-3-UC`, which preserves the critical property of the problem instance and thus proves DP-hardness of `M-3-UC`, see [CM87]. Figures 10, 11, and 12 are adapted from [CM87] with a few minor modifications.

Let the boolean formula $\varphi = (X, C)$ with variable set $X = \{x_1, x_2, \ldots, x_n\}$ and clause set $C = \{c_1, c_2, \ldots, c_m\}$ be given. Define the reduction f that maps φ to a graph G as follows. First, create two distinct vertices, v_c and v_s, and an edge connecting them. For each variable x_i, add the two vertices x_i and $\neg x_i$ representing its literals to G, and insert edges such that every pair of literal vertices corresponding to the same variable forms a triangle with the vertex v_c.

Suppose there exists a legal 3-coloring of G, and let $\{\mathrm{T}, \mathrm{F}, \mathrm{C}\}$ be the color set. Without loss of generality, let v_c be colored with C, and let v_s be colored with T. Then, only the colors T and F are available for any pair of literal vertices x_i and $\neg x_i$, see Figure 10. Thus, a legal 3-coloring of G may be regarded as a truth assignment of the variables of φ.

Finally, components for the clauses of φ are inserted. If $c_j = (\ell_{j1} \vee \ell_{j2} \vee \ell_{j3})$ is any clause of C, create a triangle with vertices t_{j1}, t_{j2}, and t_{j3}. Additionally, for each literal ℓ_{jk} with $1 \leq k \leq 3$ in c_j, there are two vertices, a_{jk} and b_{jk}, such that a_{jk} is adjacent to the corresponding literal vertex ℓ_{jk}, and b_{jk} is adjacent to the triangle vertex t_{jk}. Figure 11 shows some legally colored component for the specific clause $c_1 = (\neg x_1 \vee x_2 \vee \neg x_3)$.

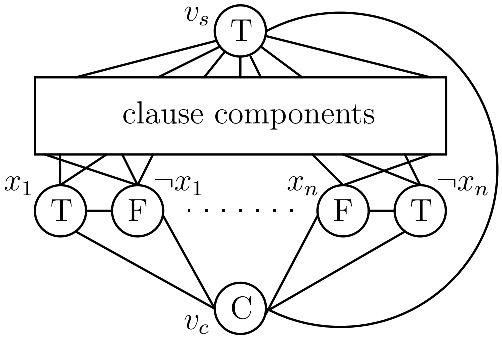

Figure 10: A legal 3-coloring of v_c, v_s, and the literal vertices of graph G

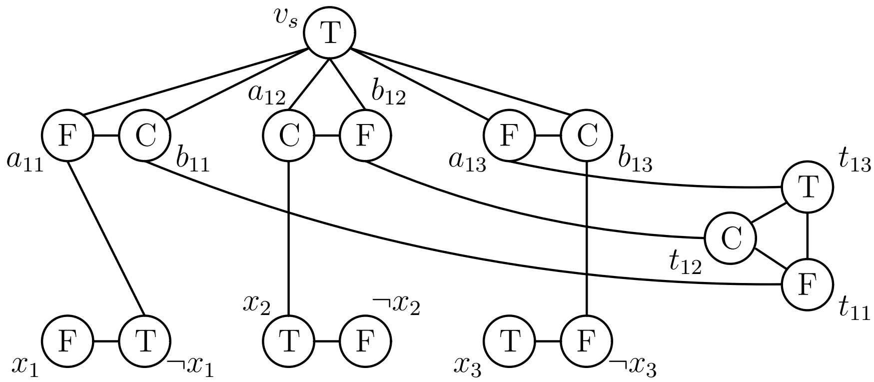

Figure 11: A legal 3-coloring of the clause component for $c_1 = (\neg x_1 \vee x_2 \vee \neg x_3)$

Note that the triangle with the vertices t_{j1}, t_{j2}, and t_{j3} for some clause c_j is legally 3-colorable if and only if not all of the so-called "fanout" vertices b_{j1}, b_{j2}, and b_{j3} are assigned color F. Coloring one of the fanout vertices of some clause c_j with C is possible only if the literal vertices are colored according to some truth assignment that satisfies the clause c_j.

This completes the reduction f mapping the boolean formula φ to the graph $G = f(\varphi)$. Figure 12 shows the graph $G = f(\varphi)$ resulting from the specific formula

$$\varphi(x_1, x_2, x_3) = (x_1 \vee x_2 \vee x_3) \wedge (\neg x_2 \vee x_3 \vee \neg x_4).$$

It can be shown that φ is satisfiable if and only if $G = f(\varphi)$ can be legally 3-colored. The proof is similar to the one proving NP-hardness for `3-Colorability` via the standard reduction from `3-SAT`; see, e.g., Stockmeyer, Garey, and John-

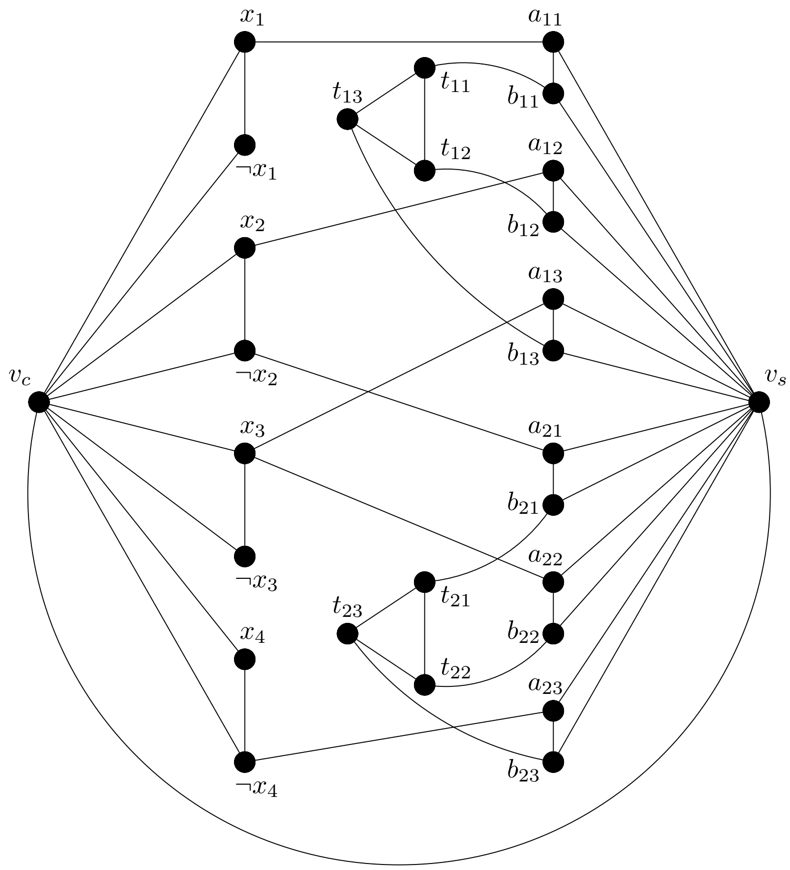

Figure 12: Graph G in the reduction Minimal-3-UNSAT $\leq_m^p$ M-3-UC

son [Sto73, GJS76, GJ79]. It remains to prove that

$$\varphi \in \texttt{Minimal-3-UNSAT} \iff G \in \texttt{Minimal-3-Uncolorability}.$$

For the direction from left to right, it is known from the claim above that the reduction f will transform any unsatisfiable formula φ into a graph G that does not have a legal 3-coloring. Analyzing the various possibilities of removing a vertex from G (for example, some literal vertex x_i or $\neg x_i$), a legal 3-coloring for the graph $G - \{v\}$ has to be determined.

For the direction from right to left, note that $G \notin$ 3-Colorability implies $\varphi \notin$ 3-SAT. Removing a clause c_j from φ, the satisfiability of the resulting formula can be deduced from the 3-colorable graph $G - \{t_{j1}\}$. For the details of the proofs of the two claims above, we refer to the original paper by Cai and Meyer [CM87].

The DP-completeness of Minimal-k-Uncolorability for $k = 3$ can easily be extended to all values of $k \geq 3$. Notice that Minimal-2-Uncolorability is in P, and thus cannot be DP-complete unless the boolean hierarchy collapses. Cai and Meyer also showed DP-completeness of Minimal-3-Uncolorability when the input is restricted to planar graphs, or to graphs with a maximum degree of five.

5 Exact Domatic Number Problems

The domatic number problem is the problem of partitioning the vertex set $V(G)$ into a maximum number of disjoint dominating sets. This number, denoted by $\delta(G)$, is called the domatic number of G. The domatic number problem arises in various real-world scenarios. For example, it is related to the tasks of distributing resources in a computer network or of locating facilities in a communication network; see, e.g., [FHK00, RR04a] for details. The domatic number problem and the closely related problem of finding a minimum dominating set in a given graph have been thoroughly studied. To name just a few papers, see, e.g., [CH77, Far84, Bon85, KS94, HT98, FHK00, RR04a, RR05, RRSY06].

Definition 12 (Domatic Number Problem). For any graph G, a *dominating set of* G is a subset $D \subseteq V(G)$ such that each vertex $u \in V(G) - D$ is adjacent to some vertex $v \in D$. Let $\delta(G)$ denote the *domatic number of* G, i.e., the maximum number of disjoint dominating sets. For each k, define the problem

$$k\text{-DNP} = \{G \mid G \text{ is a graph with } \delta(G) \geq k\}.$$

It is known that 3-DNP is NP-complete, whereas 2-DNP is in P; see Garey and Johnson [GJ79].

We now define the exact versions of domatic number problems.

Definition 13 (Exact Domatic Number Problems). Let M_k be a set that consists of k integers, and let t be a positive integer. Define

$$\text{Exact-}M_k\text{-DNP} = \{G \mid G \text{ is a graph with } \delta(G) \in M_k\},$$
$$\text{Exact-}t\text{-DNP} = \{G \mid G \text{ is a graph with } \delta(G) = t\}.$$

5.1 A General Framework for Dominating Set Problems

In order to investigate exact domatic number problems, we adopt Heggernes and Telle's general, uniform approach to define graph problems by partitioning the vertex set of a graph into generalized dominating sets [HT98]. These are subsets of the vertex set of a given graph, parameterized by two sets of nonnegative integers, σ and ρ, which restrict the number of neighbors for each vertex in the partition. Let $\mathbb{N} = \{0, 1, 2, \ldots\}$ denote the set of nonnegative integers, and let $\mathbb{N}^+ = \{1, 2, 3, \ldots\}$ denote the set of positive integers.

Definition 14 (Heggernes and Telle [HT98]). Let G be a given graph, let $\sigma \subseteq \mathbb{N}$ and $\rho \subseteq \mathbb{N}$ be given sets, and let $k \in \mathbb{N}^{+}$. Let $N(v) = \{w \in V(G) \mid \{v, w\} \in E(G)\}$ be the neighborhood of any vertex v in G.

1. A subset $U \subseteq V(G)$ of the vertices of G is said to be a (σ, ρ)*-set* if and only if
   - for each $u \in U$, $||N(u) \cap U|| \in \sigma$, and
   - for each $u \notin U$, $||N(u) \cap U|| \in \rho$.
2. A (k, σ, ρ)*-partition of* G is a partition of $V(G)$ into k pairwise disjoint subsets $V_1, V_2, \ldots, V_k$ such that V_i is a (σ, ρ)-set for each i, $1 \leq i \leq k$.
3. Define the problem

$$(k, \sigma, \rho)\text{-}\texttt{Partition} = \{G \mid G \text{ is a graph that has a } (k, \sigma, \rho)\text{-partition}\}.$$

Note that $(k, \{0\}, \mathbb{N})$-`Partition` is nothing other than k-`Colorability`, and $(k, \mathbb{N}, \mathbb{N}^{+})$-`Partition` is nothing other than k-`DNP`. This observation is illustrated by the following example. Note further that $(k, \{0\}, \mathbb{N})$-`Partition` is a minimum problem, whereas $(k, \mathbb{N}, \mathbb{N}^{+})$-`Partition` is a maximum problem.

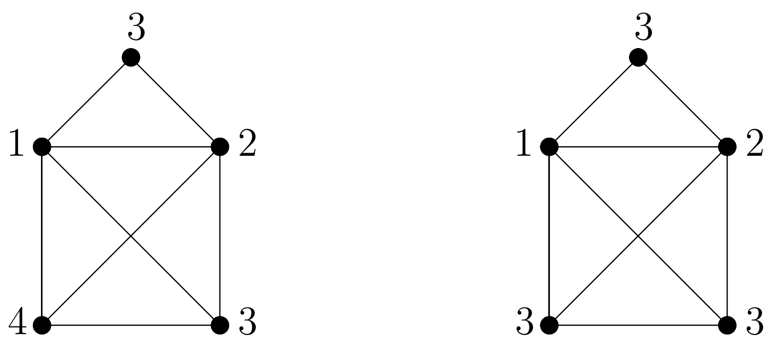

Figure 13: $(4, \{0\}, \mathbb{N})$-`Partition` (left) and $(3, \mathbb{N}, \mathbb{N}^{+})$-`Partition` (right)

Example 1 (Generalized Dominating Sets). Figure 13 shows two copies of some graph G with five vertices. Vertices labeled by the same number belong to the same (σ, ρ)-set, where either $\sigma = \{0\}$ and $\rho = \mathbb{N}$ (i.e., k-`Colorability`), or $\sigma = \mathbb{N}$ and $\rho = \mathbb{N}^{+}$ (i.e., k-`DNP`).

According to the partition into (σ, ρ)-sets shown on the left-hand side of Figure 13, G is in $(4, \{0\}, \mathbb{N})$-`Partition`. That is, G is a 4-colorable graph and the partition indicated corresponds to the four color classes of G.

In contrast, the partition into (σ, ρ)-sets on the right-hand side of Figure 13 shows that G is in $(3, \mathbb{N}, \mathbb{N}^+)$-`Partition`. That is, G has a domatic number of at least 3.

5.2 Summary of Results and Proof Ideas

Heggernes and Telle [HT98] obtained the NP-completeness results for the problems (k, σ, ρ)-`Partition` that are shown in Table 1. Here is the key: Table 1 gives the smallest value of k for which (k, σ, ρ)-`Partition` is NP-complete, where

- "∞" means that this problem is efficiently solvable for all values of k;
- a superscript "$^+$" indicates a *maximum problem*: For all $k \geq 1$,

$$(k+1, \sigma, \rho)\text{-}\mathtt{Partition} \subseteq (k, \sigma, \rho)\text{-}\mathtt{Partition};$$

  and

- a superscript "$^-$" indicates a *minimum problem*: For all $k \geq 1$,

$$(k, \sigma, \rho)\text{-}\mathtt{Partition} \subseteq (k+1, \sigma, \rho)\text{-}\mathtt{Partition}.$$

σ \ ρ	$\mathbb{N}$	$\mathbb{N}^+$	$\{1\}$	$\{0,1\}$
$\mathbb{N}$	∞^-	3^+	2	∞^-
$\mathbb{N}^+$	∞^-	2^+	2	∞^-
$\{1\}$	2^-	2	3	3^-
$\{0,1\}$	2^-	2	3	3^-
$\{0\}$	3^-	3	4	4^-

Table 1: NP-completeness for the problems (k, σ, ρ)-`Partition`

We now define the exact versions of generalized dominating set problems.

Definition 15. Define `Exact-`(k, σ, ρ)-`Partition`, the *exact version of the problem* (k, σ, ρ)-`Partition`, to be either

- (k, σ, ρ)-`Partition` $\cap$ $\overline{(k-1, \sigma, \rho)\text{-}\mathtt{Partition}}$ if (k, σ, ρ)-`Partition` is a minimum problem and $k \geq 2$, or
- (k, σ, ρ)-`Partition` $\cap$ $\overline{(k+1, \sigma, \rho)\text{-}\mathtt{Partition}}$ if (k, σ, ρ)-`Partition` is a maximum problem and $k \geq 1$.

σ \ ρ	$\mathbb{N}$	$\mathbb{N}^+$	$\{0,1\}$
$\mathbb{N}$	∞	2 | 5	∞
$\mathbb{N}^+$	∞	1 | 3	∞
$\{1\}$	2 | 5	—	3 | ?
$\{0,1\}$	2 | 5	—	3 | ?
$\{0\}$	3 | 4	—	4 | ?

Table 2: DP-completeness for the problems `Exact-`(k,σ,ρ)`-Partition`

Note that all `Exact-`(k,σ,ρ)`-Partition` problems are in DP. Note further that `Exact-`$(k,\{0\},\mathbb{N})$`-Partition` is nothing other than `Exact-`k`-Colorability`, and `Exact-`$(k,\mathbb{N},\mathbb{N}^+)$`-Partition` is nothing other than `Exact-`k`-DNP`.

Table 2 gives the best values of "$j \mid k$" for which it is known that the problem `Exact-`(k,σ,ρ)`-Partition` is "(NP-complete or coNP-complete) | DP-complete." Again, "∞" means that this problem is efficiently solvable for all values of k. Here, a dash "—" indicates that this problem is neither a maximum nor a minimum problem and thus is not considered.

Except the DP-completeness of `Exact-`$(k,\{0\},\mathbb{N})$`-Partition`, which is presented here—using different notation—as Theorem 9 in Section 3 (see [Rot03]), all DP-completeness results in Table 2 are due to Riege and Rothe [RR04a]. We state the results from Table 2 in Theorem 16 below and provide the proof ideas. We do not attempt to give full, detailed proofs, though, referring to the original source [RR04a] instead.

Theorem 16 (Riege and Rothe [Rot03]).

1. *For each* $i \geq 5$*,* `Exact-`i`-DNP` $=$ `Exact-`$(i,\mathbb{N},\mathbb{N}^+)$`-Partition` *is* DP*-complete. In contrast,* `Exact-2-DNP` $=$ `Exact-`$(2,\mathbb{N},\mathbb{N}^+)$`-Partition` *is* coNP*-complete.*
2. *For each* $i \geq 3$*,* `Exact-`$(i,\mathbb{N}^+,\mathbb{N}^+)$`-Partition` *is* DP*-complete. In contrast,* `Exact-`$(1,\mathbb{N}^+,\mathbb{N}^+)$`-Partition` *is* coNP*-complete.*
3. *For each* $i \geq 5$*,* `Exact-`$(i,\{0,1\},\mathbb{N})$`-Partition` *is* DP*-complete. In contrast,* `Exact-`$(2,\{0,1\},\mathbb{N})$`-Partition` *is* NP*-complete.*
4. *For each* $i \geq 5$*,* `Exact-`$(i,\{1\},\mathbb{N})$`-Partition` *is* DP*-complete. In contrast,* `Exact-`$(2,\{1\},\mathbb{N})$`-Partition` *is* NP*-complete.*

All proofs of the four claims of Theorem 16 essentially follow the same idea. Starting from two instances of an NP-complete problem, two graphs G_1 and G_2 corresponding to the underlying (k,σ,ρ)`-Partition` problem are generated via a polynomial-time many-one reduction. These graphs G_1 and G_2 are then merged

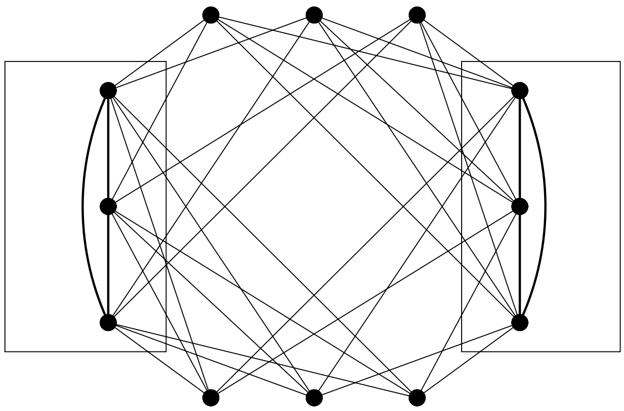

Figure 14: Gadget for proving `Exact`-5-`DNP` DP-complete

such that their parameters corresponding to the property being considered (for example, their domatic numbers in the first case) are added up. The gadgets used to accomplish this in the four different cases are presented in Figures 14, 15, and 16, which have been taken from [RR04a] in slightly modified form. All details of the proofs sketched here can be found in [RR04a].

The proof of the first part of Theorem 16 uses the gadget shown in Figure 14 to provide a reduction from `3-Colorability` that satisfies the hypothesis (2.2) of Wagner's Lemma 7. The construction in Figure 14 extends Kaplan and Shamir's reduction from `3-Colorability` to 3-`DNP` with useful properties [KS94], see also [RR04a].

The proof of the second part of Theorem 16 uses the gadget shown in Figure 15 to provide a reduction from `NAE-3-SAT` that satisfies the hypothesis (2.2) of Wagner's Lemma 7. The problem `NAE-3-SAT` ("not-all-equal satisfiability for boolean formulas with three literals per clause") asks whether a given boolean formula φ can be satisfied such that in none of the clauses of φ all literals are true. Schaefer proved that `NAE-3-SAT` is NP-complete [Sch78]. The construction in Figure 15 is inspired by Heggernes and Telle's reduction from `NAE-3-SAT` to $(2, \mathbb{N}^+, \mathbb{N}^+)$-`Partition`, see [HT98] and also [RR04a].

The proof of the third part of Theorem 16 uses a reduction from `1-3-SAT` that satisfies the hypothesis (2.2) of Wagner's Lemma 7. The problem `1-3-SAT` ("one-in-three satisfiability") asks whether, given a boolean formula φ, there exists a subset T of the literals of φ with $||T \cap C_i|| = 1$ for each clause C_i. Schaefer proved that `1-3-SAT` is NP-complete, even if all literals in the given boolean formula are

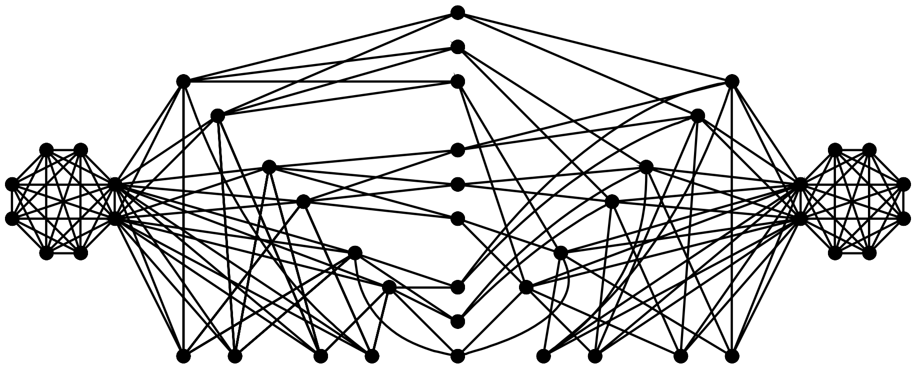

Figure 15: Gadget for proving `Exact-`$(3, \mathbb{N}^+, \mathbb{N}^+)$`-Partition` DP-complete

positive [Sch78].

Figure 16 shows this construction, which is based on Heggernes and Telle's reduction from `1-3-SAT` to $(2, \{0,1\}, \mathbb{N})$`-Partition`, see [HT98]. The symbol $\oplus$ in Figure 16 denotes the *join operation on graphs*, i.e., for any two graphs G_1 and G_2, $G_1 \oplus G_2$ is the graph with vertex set

$$V(G_1 \oplus G_2) = V(G_1) \cup V(G_2)$$

and edge set

$$E(G_1 \oplus G_2) = E(G_1) \cup E(G_2) \cup \{\{a, b\} \mid a \in V(G_1) \text{ and } b \in V(G_2)\}.$$

The proof of the fourth part of Theorem 16 is obtained by suitably modifying the proof of the third part of Theorem 16.

Generalizing the results on exact generalized dominating set problems from Theorem 16, Riege and Rothe [RR04a] obtained completeness results in the higher levels of the boolean hierarchy. We state this generalization for the problem `Exact-`M_k`-DNP` only, where $M_k = \{4k+1, 4k+3, \ldots, 6k-1\}$, in Theorem 17 below. Analogously, the completeness results for `Exact-`(k, σ, ρ)`-Partition` given in the second, third, and fourth part of Theorem 16 can be lifted to the higher levels of the boolean hierarchy over NP.

Theorem 17 (Riege and Rothe [Rot03]). *For* $M_k = \{4k+1, 4k+3, \ldots, 6k-1\}$*, the problem* `Exact-`$M_k$`-DNP` *is* $\mathrm{BH}_{2k}(\mathrm{NP})$*-complete.*

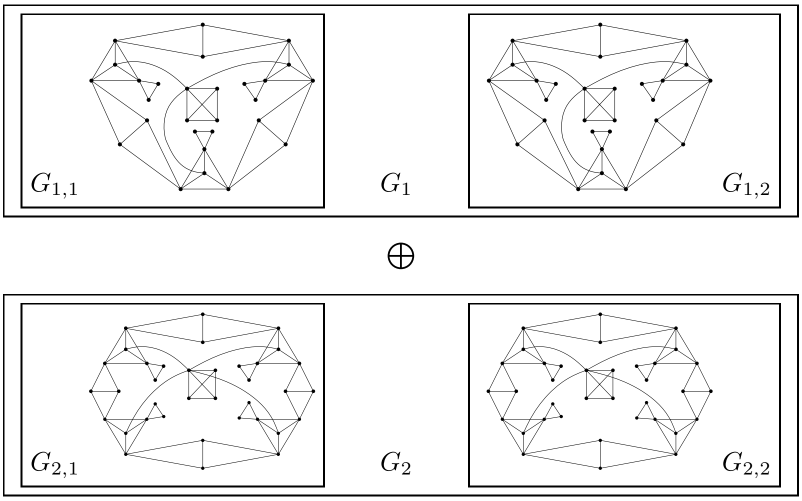

Figure 16: Reduction to prove Exact-$(5, \{0, 1\}, \mathbb{N})$-Partition DP-complete

Finally, define the following variants of the domatic number problem:

$$\texttt{DNP}_{\texttt{odd}} = \{G \mid G \text{ is a graph such that } \delta(G) \text{ is odd}\},$$
$$\texttt{DNP}_{\texttt{equ}} = \{\langle G, H\rangle \mid G \text{ and } H \text{ are graphs with } \delta(G) = \delta(H)\},$$
$$\texttt{DNP}_{\texttt{leq}} = \{\langle G, H\rangle \mid G \text{ and } H \text{ are graphs with } \delta(G) \leq \delta(H)\}.$$

Theorem 18 (Riege and Rothe [Rot03]). *The problems* $\texttt{DNP}_{\texttt{odd}}$*,* $\texttt{DNP}_{\texttt{equ}}$*, and* $\texttt{DNP}_{\texttt{leq}}$ *each are* Θ_2^p*-complete.*

6 Conclusions and Open Questions

This survey paper has presented some of the results that were inspired by Wagner's general technique [Wag87] to prove completeness in the levels of the boolean hierarchy over NP and in Θ_2^p, the class of problems solvable via parallel access to NP. In particular, Θ_2^p-completeness results were obtained for a variety of natural problems arising in computational politics [HHR97a, RSV03, HH00, HSV05] and for problems related to certain approximation heuristics for hard graph problems [HR98, HRS06, HHR97b]. In addition, Wagner's technique was useful to prove Θ_2^p-hardness of MEE, the minimum equivalent expression problem, see Hemaspaandra and Wechsung [HW97, HW02].

Turning to completeness in the levels of the boolean hierarchy, Theorem 9 in Section 3 answered a question raised by Wagner in [Wag87]: It is DP-complete to decide whether or not a given graph can be colored with exactly four colors.

We have sketched Guruswami and Khanna's clever reduction [GK00] that is central to this proof, and we have shown how this reduction can be employed by Wagner's technique to prove Theorem 9.

In Section 4, we presented Cai and Meyer's beautiful result that the prominent problem `Minimal-3-Uncolorability` is DP-complete [CM87]. It should be stressed here that it is usually very difficult to transfer known NP-completeness results to DP-completeness results for the corresponding critical problems. Papadimitriou and Yannakakis [PY84] note: "*We have not been able to show that [...] any of the critical problems is* DP*-complete. This difficulty seems to reflect the extremely delicate and deep structure of critical problems—too delicate to sustain any of the known reduction methods. One way to understand this is that critical graphs is usually the object of hard theorems.*" The crucial point is that polynomial-time many-one reductions from one problem to another do not preserve criticality in general. For this reason, only very few critical problems are known to be DP-complete up to date.

Finally, Section 5 studied various versions of the exact domatic number problem. In particular, Theorem 16 says that `Exact-5-DNP` is DP-complete. In contrast, `Exact-2-DNP` is coNP-complete, and thus this problem cannot be DP-complete unless the boolean hierarchy collapses. For $i \in \{3, 4\}$, the question of whether or not the problems `Exact-`i`-DNP` are DP-complete remains open. To close this gap, it would be enough to find a reduction from some suitable NP-complete problem to the exact domatic number problem that yields graphs having a domatic number other than three.

In addition, we have studied the exact versions of Heggernes and Telle's generalized dominating set problems [HT98], denoted by `Exact-`(k, σ, ρ)`-Partition`, where the parameters σ and ρ specify the number of neighbors that are allowed for each vertex in the partition. Theorem 16 presented DP-completeness results for a number of such problems that are summarized in Table 2, which gives the best values of k for which the problems `Exact-`(k, σ, ρ)`-Partition` are known to be DP-complete. This value of k is not yet optimal in some cases. For example, as stated in Theorem 16, `Exact-`$(5, \{0, 1\}, \mathbb{N})$`-Partition` is DP-complete and `Exact-`$(2, \{0, 1\}, \mathbb{N})$`-Partition` is NP-complete. What about the complexity of `Exact-`$(i, \{0, 1\}, \mathbb{N})$`-Partition` for $i \in \{3, 4\}$? It would also be interesting to obtain DP-completeness results for those cases in Table 2 that currently have only question marks.

Acknowledgments

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