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Balance in Systems of Finite Sets with Applications[1]

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Abstract: An extension of balance notion from the theory of signed graphs to the case of finite sets systems is presented. For a finite set T, a subset $S \subseteq T$ and a family $\mathcal{F}$ of subsets of T we denote by $\delta_m(S|\mathcal{F})$ respectively $\delta_M(S|\mathcal{F})$ the minimum/maximum number of changes (addition or deletion of elements), without repetition, which transforms S into a set from $\mathcal{F}$.

We are especially interested in the particular case in which $\mathcal{F}$ is the group $<X_1, \ldots, X_n>$ generated by a family of subsets $X_1, \ldots, X_n \subseteq T$ with symmetric difference operation. The obtained results are applied to the theory of signed graphs.

Key Words: balancing signed graphs
Category: G.2.2

1 Definition and Notations

1.1

Let T be a finite set and $T^{(*)} = \{X | X \subseteq T\}$ be the set of its subsets. The set $T^{(*)}$ with symmetric difference operation $\varDelta$

$$X \varDelta Y := (X - Y) \cup (Y - X) \text{ for } X, Y \in T^{(*)}$$

constitutes a commutative group $\left(T^{(*)}, \varDelta\right)$ isomorphic to $\left(\mathbb{Z}_2^{|T|}, +\right)$.

Let $X_1, \ldots, X_n \in T^{(*)}$. We denote

$$X_J := \begin{cases} \varDelta_{j \in J} X_j & \text{for } \varnothing \neq J \subseteq [n] \\ \varnothing & \text{for } J = \emptyset \end{cases}$$

and

$$< X_1, \ldots, X_n >= X_J | J \subseteq [n] \,,$$

the group generated by $X_1, \ldots, X_n$ in $\left(T^{(*)}, \varDelta\right)$.

For a set $K \subseteq \mathbb{N}$ we denote by $o(K)$ the number of odd numbers from K. For $n, k \in \{0, 1, 2, \ldots\}$ we denote by

$$n^{\underline{k}} := \begin{cases} n(n-1)(n-2)\cdots(n-k+1) & \text{for } k \geq 1 \\ 1 & \text{for } k = 0 \end{cases}$$

the falling factorial.

[1] C. S. Calude, G. Stefanescu, and M. Zimand (eds.). *Combinatorics and Related Areas. A Collection of Papers in Honour of the 65th Birthday of Ioan Tomescu.*

1.2

Let $S \in T^{(*)}$ and $\mathcal{F} \subseteq T^{(*)}$ be a family of subsets of T. We denote by

$$\begin{aligned}\delta_m (S|\mathcal{F}) &:= \min \{|S\Delta F| \ \ |F \in \mathcal{F}|\} \\ \delta_M (S|\mathcal{F}) &:= \max \{|S\Delta F| \ \ |F \in \mathcal{F}|\}\end{aligned}$$

the minimum, respectively maximum number of changes (addition or deletion of elements) without repetition, which transforms S into a set from $\mathcal{F}$.

In other words, $\delta_m (S|\mathcal{F})$ and $\delta_M (S|\mathcal{F})$ are minimum, respectively maximum Hamming distances apart from S to an element of $\mathcal{F}$.

1.3

In this paper we consider $\mathcal{F} =< X_1, \ldots, X_n >$ and we study the numbers

$$\delta_m (S| < X_1, \ldots, X_n >) \text{ and } \delta_M (S| < X_1, \ldots, X_n >)$$

which will be called the minimum (respectively maximum) unbalanced index.

Every set $S \in< X_1, \ldots, X_n >$ will be called a balanced set.

2 The main result

Lemma 1. *Let T be a finite set and $S, X \subseteq T$. We have*

$$\text{(i) } \delta_m(S\,|< X >) \leq |S - X| + |X|\,/2 - \frac{1}{2} o\left(\{|X|\}\right) \tag{1}$$

$$\text{(ii) } \delta_M(S\,|< X >) \geq |S - X| + |X|\,/2 + \frac{1}{2} o\left(\{|X|\}\right). \tag{2}$$

Proof. We have

$$\begin{aligned}|S| &= |S - X| + |S \cap X| \\ |S\Delta X| &= |S - X| + |X| - |S \cap X| \\ \delta_m (S| < X >) &= \min \{|S|\,, |S\Delta X|\} \\ \delta_M (S| < X >) &= \max \{|S|\,, |S\Delta X|\}\end{aligned}$$

and the Lemma follows.□

Theorem 2. *Let T be a finite set and $S, X_1, \ldots, X_n \subseteq T$. We have:*

$$\text{(i) } \begin{aligned}&\delta_m (S| < X_1, \ldots, X_n >) \leq \left| S - \underset{i \in [n]}{\cup} X_i \right| + \left| \underset{i \in [n]}{\cup} X_i \right| /2 - \\ &-\frac{1}{2} o\left(\left\{\left| X_j - \underset{1 \leq t < j}{\cup} X_t \right| \ \ |j \in [n]\right\}\right)\end{aligned} \tag{3}$$

$$\text{(4)} \quad \text{(ii)} \quad \delta_M\left(S| < X_1, \ldots, X_n >\right) \geq \left|S - \underset{i\in[n]}{\cup} X_i\right| + \left|\underset{i\in[n]}{\cup} X_i\right|/2 + \\ +\frac{1}{2}o\left(\left\{\left|X_j - \underset{1\leq t<j}{\cup} X_t\right| \ |j \in [n]\right\}\right).$$

Proof. The two inequalities have similar proofs. We shall consider only the first.

We use induction on n and denote by $\mathcal{I}(n)$ the inductive hypothesis.

$\mathcal{I}(1)$ is true according to Lemma 1.

We shall prove that $\mathcal{I}(1), \mathcal{I}(n-1)$ imply $\mathcal{I}(n). (n \geq 2)$. We have

$$\delta_m\left(S| < X_1, \ldots, X_n >\right) = \min\left\{|S \ \triangle \ \underset{i\in I}{\Delta} X_i| \ |I \subseteq [n]\right\} =$$

$$= \min\left\{\delta_m\left(S| < X_1, \ldots, X_{n-1} >\right), \delta_m\left(S \ \triangle \ X_n| < X_1, \ldots, X_{n-1} >\right)\right\} \leq$$

$$\leq \min\left\{\left|S - \underset{i\in[n-1]}{\cup} X_i\right|, \left|(S \ \triangle \ X_n) - \underset{i\in[n-1]}{\cup} X_i\right|\right\} + \left|\underset{i\in[n-1]}{\cup} X_i\right|/2 -$$

$$-\frac{1}{2}o\left(\left\{\left|X_j - \underset{1\leq k<j}{\cup} X_k\right| |j \in [n-1]\right\}\right) =$$

$$= \min\left\{\left|S - \underset{i\in[n-1]}{\cup} X_i\right|, \left|\left(S - \underset{i\in[n-1]}{\cup} X_i\right) \triangle \ \left(X_n - \underset{i\in[n-1]}{\cup} X_i\right)\right|\right\} +$$

$$+\left|\underset{i\in[n-1]}{\cup} X_i\right|/2 - \frac{1}{2}o\left(\left\{\left|X_j - \underset{1\leq k<j}{\cup} X_k\right| |j \in [n-1]\right\}\right) =$$

$$= \delta_m\left(\left|S - \underset{i\in[n-1]}{\cup} X_i\right| < X_n - \underset{i\in[n-1]}{\cup} X_i >\right) + \left|\underset{i\in[n-1]}{\cup} X_i\right|/2 -$$

$$-\frac{1}{2}o\left(\left\{\left|X_j - \underset{1\leq k<j}{\cup} X_k\right| |j \in [n-1]\right\}\right) \leq$$

$$\leq \left|\left(S - \underset{i\in[n-1]}{\cup} X_i\right) - \left(X_n - \underset{i\in[n-1]}{\cup} X_i\right)\right| + \left|X_n - \underset{i\in[n-1]}{\cup} X_i\right|/2 -$$

$$-\frac{1}{2}o\left(\left\{\left|X_n - \underset{i\in[n-1]}{\cup} X_i\right|\right\}\right) + \left|\underset{i\in[n-1]}{\cup} X_i\right|/2 -$$

$$-\frac{1}{2}o\left(\left\{\left|X_j - \underset{1\leq k<j}{\cup} X_k\right| |j \in [n-1]\right\}\right) =$$

$$= \left|S - \underset{i\in[n]}{\cup} X_i\right| + \left|\underset{i\in[n]}{\cup} X_i\right|/2 - \frac{1}{2}o\left(\left\{\left|X_j - \underset{1\leq k<j}{\cup} X_k\right| |j \in [n]\right\}\right). \square$$

Remark. For $T = \underset{i\in[n]}{\cup} X_i$ we have

$$\text{(5)} \quad \text{(i)} \ \delta_m\left(S \ | \ < X_1, \ldots, X_n >\right) \leq \\ \leq |T|/2 - \frac{1}{2}o\left(\left\{\left|X_j - \underset{1\leq k<j}{\cup} X_k\right| |j \in [n]\right\}\right),$$

$$\text{(ii) } \delta_M\left(S \mid\ < X_1, \ldots, X_n >\right) \leq$$
$$\leq |T|/2 + \frac{1}{2} o\left(\left\{\left| X_j - \underset{1\leq k<j}{\cup} X_k \right| |j \in [n]\right\}\right).$$

Lemma 3. *Let A, B, C be three finite sets. We have*

(i) $$A \triangle\ B - C = (A - C) \triangle (B - C)$$

(ii) If $B \supseteq C$ then:

(8). $$|A \triangle C| = |A - B| + |(A \cap B) \triangle C| . \square$$

Theorem 4. *Let T be a finite set $S, X_1, \ldots, X_n \subseteq T$ and $I_1 \dot\cup \cdots \dot\cup I_p = [n]$ be a p-partition into nonempty parts of $[n]$. We have*

(9)

$$\text{(i) } \delta_m\left(S \mid\ < X_1, \ldots, X_n >\right) \leq$$
$$\leq \delta_m\left(S - Z| < X_1 - Z, \ldots, X_n - Z >\right) + |Z|/2 -$$
$$-\frac{1}{2} o\left(\left\{|X_{I_1}|, |X_{I_2} - X_{I_1}|, |X_{I_3} - X_{I_2} - X_{I_1}|, \ldots, \left|X_{I_p} - X_{I_{p-1}} - \cdots - X_{I_1}\right|\right\}\right)$$

(10)

$$\text{(ii) } \delta_M\left(S \mid\ < X_1, \ldots, X_n >\right) \geq$$
$$\geq \delta_M\left(S - Z| < X_1 - Z, \ldots, X_n - Z >\right) + |Z|/2 +$$
$$+\frac{1}{2} o\left(\left\{|X_{I_1}|, |X_{I_2} - X_{I_1}|, |X_{I_3} - X_{I_2} - X_{I_1}|, \ldots, \left|X_{i_p} - X_{i_{p-1}} - \cdots - X_{I_1}\right|\right\}\right)$$

where

$$X_J := \begin{cases} \underset{j\in J}{\triangle} X_j & \textit{for } \varnothing \neq J \subseteq [n] \\ \varnothing & \textit{for } J = \varnothing \end{cases}$$

and

$$Z := \underset{t\in L}{\cup} X_{I_t}.$$

Proof. For any $L \subseteq [p]$, with the notation

$$I_L := \underset{t\in L}{\cup} I_t$$

we have

$$X_{I_L} = X_{\underset{t\in L}{\cup} I_t} = \underset{j\in \underset{t\in L}{\cup} I_t}{\Delta} X_j = \underset{t\in L}{\Delta} X_{I_t}.$$

(i) We choose $K_0 \subseteq [n]$ such that $|(\mathcal{S}\Delta X_{K_0}) - Z|$ is minimum. We use the fact that

$$Z \supseteq X_{I_L}, \forall L \subseteq [p],$$
$$(\mathcal{S}\Delta X_{K_0}) \cap Z \subseteq X_{I_1} \cup \cdots \cup X_{I_t} = Z,$$

we apply Lemma 3(ii),(i), Theorem 2 and Remark(after Theorem 2)(i) and we obtain

$$\delta\left(S \mid\ < X_1, \dots, X_n >\right) = \min\left\{\left|S \Delta X_J\right|\ \mid J \subseteq [n]\right\} \leq$$
$$\leq \min\left\{\left|S \Delta X_{K_0} \Delta X_j\right|\ \mid j \subseteq [n]|\right\} \leq \min\left\{\left|S \Delta X_{K_0} \Delta X_{I_L}\right|\ \mid L \subseteq [p]\right\} =$$
$$= \min\left\{\left|(S \Delta X_{K_0}) - Z\right| + \left|((S \Delta X_{K_0}) \cap\ Z)\, \Delta X_{I_L}\right| |L \subseteq [p]\right\} =$$
$$\left|(S \Delta X_{K_0}) - Z\right| + \min\left\{\left|((S \Delta X_{K_0}) \cap Z)\, \Delta X_{I_L}\right|\ \mid L \subseteq [p]\right\} =$$
$$= \left|(S - Z)\, \Delta\, (X_{K_0} - Z)\right| + \delta_m\left((S \Delta X_{K_0}) \cap Z| < X_{I_1}, \dots, X_{I_p} >\right) =$$
$$\left|(S - Z)\, \Delta \left(\underset{t \in K_0}{\Delta} X_t - Z\right)\right| + \delta_m\left((S \Delta X_{K_0}) \cap Z| < X_{I_1}, \dots, X_{I_p} >\right) =$$
$$= \left|(S - Z)\, \Delta \underset{t \in K_0}{\Delta} (X_t - Z)\right| + \delta_m\left((S \Delta X_{K_0}) \cap Z| < X_{I_1}, \dots, X_{I_p} >\right) \leq$$
$$\leq \delta_m\left(S - Z| < X_{I_1} - Z, \dots, X_{I_p} - Z >\right) + |Z|/2 -$$
$$-\frac{1}{2} o\left(\left\{\left|X_{I_1}\right|, \left|X_{I_2} - X_{I_1}\right|, \left|X_{I_3} - X_{I_2} - X_{I_1}\right|, \dots, \left|X_{i_p} - X_{i_{p-1}} - \dots - X_{I_1}\right|\right\}\right).$$

(ii) We choose $K_0 \subseteq [n]$ such that $\left|(S \Delta X_{K_0}) - Z\right|$ is maximum and the proof follows the same way.□

Remark. For $p = n$ and $I_t = \{t\}, t \in [n]$ we have

$$X_{I_t} = X_t$$
$$Z = \underset{i \in [n]}{\cup} X_i$$

(11) (i) $\delta_m\left(S - Z| < X_{I_1} - Z, \dots, X_n - Z >\right) = \delta_m\left(S - Z| < \varnothing, \dots, \varnothing >\right) = \left|S - \underset{i \in [n]}{\cup} X_i\right|$

(12) (ii) $\delta_M\left(S - Z| < X_{I_1} - Z, \dots, X_n - Z >\right) = \delta_M\left(S - Z| < \varnothing, \dots, \varnothing >\right) = \left|S - \underset{i \in [n]}{\cup} X_i\right|$.

Thus, we obtain Theorem 2 from Theorem 4.

3 Applications to the Theory of signed graphs

A signed graph is a simple graph $G = (V, E)$ with an edge set bipartition $E = E^- \cup E^+$. We say that the edges from E^- are negative and those from E^+ are positive. We denote by $G^- := (V, E^-)$ and $G^+ := (V, E^+)$ the spanning subgraphs of G with edge set E^- respectively E^+. A subgraph $G' \leq G$ is said to be negative if it contains an odd number of negative edges and it is positive otherwise. A signed graph is called balanced if each of its cycles is positive or, by Cartwright and Harary's characterization [Cartwright and Harary 1956], if and only if the vertex set may be partitioned in two subsets (one of them may be empty) so that each positive edge has its ends in the same subset and each negative edge has its ends in different subsets. The unbalanced index $\delta(G)$ of a signed graph G is the minimum number of sign-changes of edges which transforms G into a balanced graph.

3.1

Let $G = (V, E)$ be a signed graph with $V = \{1, 2, 3, \ldots, n\}$.

For $\varnothing \neq A, B \subseteq V$ we denote

$E[A, B] :=$ the edges of G which have one end in A and one end in B

$\overline{A} := V - A$

$G[A] :=$ the subgraph induced in G by the vertices of A

$[n] := \{1, 2, 3, \ldots n\}$

$f(n) := \lfloor (n-1)^2 / 4 \rfloor$

$T := E(G)$ the edge set of G

$X_i := E[\{i\}, V - \{i\}]$, the edge set of G that contains i

$$X_I := \begin{cases} \underset{i \in I}{\Delta} X_i & \text{for } \varnothing \neq I \subseteq V \\ \varnothing & \text{for } I = \varnothing \end{cases}$$

$S := E^-(G)$ the negative edge set of G.

It is easy to show the following estimation of the unbalanced index of G

$$\delta(G) = \delta_m(S| < X_{I_1}, \ldots, X_n >) = \min\{|S \Delta X_I| \, |I \subseteq V|\}.$$

We shall denote $\delta_m(G)$ in place of $\delta(G)$ and we denote analogously by

$$\delta_M(G) := \delta_M(S| < X_1, \ldots, X_n >)$$

the maximum number of sign-changes of edges (without repetition) which transforms G into a balanced graph.

Let $V_1 \dot{\cup} \ldots \dot{\cup} V_p = V$ be a p-partition with nonempty parts of V. We denote

$$\begin{aligned} G_t &:= G[V_t], \text{ for } t \in [p] \\ X_{V_t} &:= \underset{j \in V_t}{\Delta} X_j, \text{ for } t \in [p] \\ Z &:= \bigcup_{t \in [p]} X_{V_t}. \end{aligned}$$

We consider the signed graphs $G_t; t \in [p]$ with the induced bipartition: $E^-(G_t) = E(G_t) \cap E^-(G)$. It is easy to prove the following equalities

$$X_{V_t} = E\left[V_t, \overline{V}_t\right] \text{ for } t \in [p]$$

$$Z = E(G) - E(G_1) - \cdots - E(G_p)$$

$$X_{V_t} - X_{V_{t-1}} - X_{V_{t-2}} - \cdots - X_{V_1} = E[V_t, V_{t+1} \cup \cdots \cup V_p], \text{ for } t \in [p]$$

$$\delta_m(S - Z| < X_1 - Z, \ldots, X_n - Z >) = \delta_m(G_1) + \cdots + \delta_m(G_p)$$

$$\delta_M(S - Z| < X_1 - Z, \ldots, X_n - Z >) = \delta_M(G_1) + \cdots + \delta_M(G_p).$$

Lemma 5. *For any* $n_1, \ldots, n_p \in \mathbb{N}$ *we have*

$$\text{(i)} \lfloor o(\{n_1, \ldots, n_p\})/2 \rfloor = \lfloor (n_1 + \cdots + n_p)/2 \rfloor - \lfloor n_1/2 \rfloor - \cdots - \lfloor n_p/2 \rfloor; \tag{13}$$

$$\begin{aligned} &\text{(ii) } \lfloor o(\{n_1, \ldots n_p\})/2 \rfloor = \\ &\qquad = o(\{n_1(n_2 + \cdots + n_p), n_2(n_3 + \cdots + n_p), \ldots, n_{p-1}n_p\}). \square \end{aligned} \tag{14}$$

Lemma 6. *For any* $n \in \mathbb{N}$ *we have*

$$\text{(i) } f(n) = \frac{1}{2}\binom{n}{2} - \frac{1}{2}\lfloor n/2 \rfloor; \tag{15}$$

$$\text{(ii) } f(n+1) = \frac{1}{2}\binom{n}{2} + \frac{1}{2}\lfloor n/2 \rfloor. \square \tag{16}$$

Theorem 7. *Let* $G = (V, E)$ *be a signed graph and* $V_1 \dot{\cup} \cdots \dot{\cup} V_p = V$ *a* p *-partition with nonempty parts of* V. *We have*

$$\text{(i) } \delta_m(G) - |E(G)|/2 \leq \tag{17}$$

$$\sum_{t \in [p]} (\delta_m(G_t) - |E(G_t)|/2) - \frac{1}{2} o(\{|E[V_t, V_{t+1} \cup \cdots \cup V_p]| \ |1 \leq t < p|\})$$

$$\text{(ii) } \delta_M(G) - |E(G)|/2 \geq \tag{18}$$

$$\sum_{t \in [p]} (\delta_M(G_t) - |E(G_t)|/2) + \frac{1}{2} o(\{|E[V_t, V_{t+1} \cup \cdots \cup V_p]| \ |1 \leq t < p\})$$

where $G_t = G[V_t]$ *for* $t \in [p]$.

Proof. The two inequalities have similar proofs. We shall consider only the first.

According to Theorem 2(i) and the preceding estimations we have

$$\delta_m(G) \leq (\delta_m(G_1) + \cdots + \delta_m(G_p)) + (|E(G)| - |E(G_1)| - \cdots - |E(G_p)|)/2- \\ -\frac{1}{2} o\left(\{|E[V_t, V_{t+1} \cup \cdots \cup V_p]|\, |1 \leq t < p|\}\right)$$

and (i) follows. □

Theorem 8. *Let $G = (V, E) \sim K_n$ be a complete signed graph and $V_1 \dot{\cup} \cdots \dot{\cup} V_p = V$ be a p-partition with nonempty parts of V. We have*

$$\text{(i) } \delta_m(G) - f(n) \leq \sum_{t\in[p]} (\delta_m(G_t) - f(n_t)), \tag{19}$$

$$\text{(ii) } \delta_M(G) - f(n+1) \geq \sum_{t\in[p]} (\delta_M(G_t) - f(n_t + 1)), \tag{20}$$

where $G_t = G[V_t]$, $n_t = |V_t|$ for $t \in [p]$ and $n = |V|$.

Proof. The two inequalities have similar proofs. We shall consider only the first.

We have

$|E(G)| = \binom{n}{2}$

$|E(G_t)| = \binom{n_t}{2}$, for $t \in [p]$

$|E[V_t, V_{t+1} \cup \cdots \cup V_p]| = n_t(n_{t+1} + \cdots + n_p)$, for $t \in \{1, 2, 3, \ldots, p-1\}$

$n = n_1 + \cdots + n_p$.

According to Theorem 7(i) we have

$$\delta_m(G) - \left(\frac{1}{2}\binom{n}{2} - \frac{1}{2}\lfloor n/2 \rfloor\right) \leq$$

$$\leq \sum_{t\in[p]} \left(\delta_m(G_t) - \left(-\frac{1}{2}\binom{n_t}{2} - \frac{1}{2}\lfloor n_t/2 \rfloor\right)\right) + \frac{1}{2}\left(\lfloor n/2 \rfloor - \sum_{t\in[p]} \lfloor n_t/2 \rfloor\right) - \\ -\frac{1}{2} o\left(\{n_1(n_2 + \cdots + n_p), n_2(n_3 + \cdots + n_p), \ldots, n_{p-1}n_p\}\right)$$

and by Lemmas 5 and 6(i) we obtain the inequality (i) from the theorem. □

Corollary 9. *Let $G-(V, E)$ be a signed graph, $V_1 \dot{\cup} \cdots \dot{\cup} V_p = V$ be a p -partition with nonempty part of V and $G_t := G[V_t], t \in [p]$ be the induced signed graphs. If G_t is balanced for $t \in [p]$ then we have*

$$\text{(i) } \delta_m(G) \leq \tag{21}$$

$$\leq \frac{1}{2}\left(|E(G)| - \sum_{t\in[p]} |E(G_t)|\right) - \frac{1}{2} o\left(\left\{\left|E\left[V_t, V_{t+1} \dot\cup \cdots \dot\cup V_p\right]\right| |1 \leq t < p\right\}\right)$$

$$\text{(ii) } \delta_M(G) \geq \tag{22}$$

$$\geq \frac{1}{2}\left(|E(G)| - \sum_{t\in[p]} |E(G_t)|\right) + \frac{1}{2} o\left(\left\{\left|E\left[V_t, V_{t+1} \dot\cup \cdots \dot\cup V_p\right]\right| |1 \leq t < p\right\}\right).$$

Proof. We aply the Theorem 7 for $\delta(G_t) = 0,\ t \in [p]$ and the Corollary follows. □

Remark. If we ignore in (21) the term

$$\frac{1}{2} o\left(\left\{\left[V_t, V_{t+1} \dot\cup \cdots \dot\cup V_p\right] |1 \leq t < p\right\}\right)$$

we have

$$\delta_m(G) \leq \frac{1}{2}\left(|E(G)| - \sum_{t\in[p]} |E(G_t)|\right), \tag{23}$$

an inequality obtained in 1981 by J. Akiyama, D. Avis, V. Chvátal and H. Era, [Akiyama et al. 1981].

Corollary 10. *Let $G = (V, E) \sim K_n$ be a complete signed graph with $n = |V|$. We have*

$$\text{(i) } \delta_m(G) \leq f(n) \tag{24}$$

(I. Tomescu, 1973, [Tomescu 1973])

$$\text{(i) } \delta_M(G) \geq f(n+1). \tag{25}$$

Proof. We apply the Theorem 8 for $p = n, V_t = [t] .t \in [n]$ and we have
$\delta_m(G_t) = \delta_M(G_t) = 0$
$f(|V_t|) = f(1) = 0$
$f(|V_t| + 1) = f(2) = 0.$
(i) $\delta_m(G) - f(n) \leq -nf(1) = 0$
(ii) $\delta_M(G) - f(n+1) \geq -nf(2) = 0.$□

Remark. (i) The inequality (24) was proposed without proof in 1959 by R. Abelson and M.J. Rosenberg [Abelson and Rosenberg 1958] and it was proved in 1973 by Ioan Tomescu [Tomescu 1973].

(ii) It is easy to show that in (24) equality holds if an only if $G^+ \sim K_{\lfloor n/2\rfloor\lceil n/2\rceil}$ and in (25) equality holds if and only if $G^- \sim K_{\lfloor n/2\rfloor\lceil n/2\rceil}$.

Corollary 11. *Let $G = (V, E) \sim K_n$ be a complete signed graph $\varnothing \neq V' \subseteq V$ and $G' = G[V']$ be the induced signed graph with $n = |V|$ and $n' = |V'|$.*

We have

$$\text{(i) } \delta_m(G) - f(n) \leq \delta_m(G') - f(n') \tag{26}$$

$$\text{(ii) } \delta_M(G) - f(n+1) \geq \delta_M(G') - f(n'+1). \tag{27}$$

Proof. If $V' = V$ we have equality in (26) and (27).

If $V' \neq V$ we denote $V'' = V - V', n'' = |V''|$ and $G'' = G[V'']$ the induced signed graph. We apply the Theorem 8 and Corollary 9 and we have

(i) $\delta_m(G) - f(n) \leq (\delta_m(G') - f(n')) + (\delta_m(G'') - f(n'')) \leq \delta_m(G') - f(n')$

(ii) $\delta_M(G) - f(n+1) \geq (\delta_m(G') - f(n'+1)) + (\delta_M(G'') - f(n''+1))$
$\geq \delta_M(G') - f(n'+1)$. □

Remark. If the subgraph G'' is balanced then we obtain from (26) and (27)

$$\delta_m(G) \leq f(n) - f(n') \tag{28}$$

$$\delta_M(G) \geq f(n+1) - f(n'+1). \tag{29}$$

The inequality (28) was obtained in 1976 by T. Sozansky, [Sozanski 1976].

3.2 An inequality on the maximum number of negative cycles in complete signed graphs

Let $G \sim K_n$ be a complete graph with $V(G) = \{1, 2, 3, \ldots, n\}$. We denote
$C(G) :=$ the set of cycles contained by the graph G;
$C(G; e) :=$ the set of cycles G that contain the edge $e, e \in E(G)$;
$C_k(G) :=$ the set of k-cycles of $G, k \geq 3$;
$C_k(G; e) :=$ the set of k-cycles of G that contain the edge $e, e \in E \in G, k \geq 3$;
$T := C(G)$;
$X_e := C(G; e)$ for $e \in E(G)$;
$X_F := \underset{e \in F}{\Delta} X_e =$ the set of cycles of G that contain an odd number of edges from $F, F \subseteq E(G)$.

If we interpret F as the negative edge set of G then X_F is the set of negative cycles of G.

Theorem 12. *Let $G \sim K_n$ be a complete graph. We have*

$$\max\{|X_F| \ \ |F \subseteq E(G)\} \geq \frac{1}{4} \sum_{3 \leq k \leq n} \frac{n^{\underline{k}}}{k} + \frac{1}{4} \lfloor \frac{n-1}{2} \rfloor \cdot \lfloor \frac{n+1}{2} \rfloor. \tag{29}$$

Proof. We consider the edge set of G in lexicographic order

$$E(G) = \left\{e_1, e_2, e_3, \ldots, e_{\binom{n}{2}}\right\}.$$

The first $(n-1)$ edges are incident in 1, the following $(n-2)$ edges are incident in 2 but not in 1, the following $(n-3)$ edges are incident in 3 but not in 1 and 2, etc.

According to Theorem 2(ii) we have

$$\begin{aligned} &\max\{|X_F| \mid F \subseteq E(G)\} = \delta_M\left(\varnothing| < X_{e_1}, X_{e_2}, \ldots, X_{e_{\binom{n}{2}}} >\right) \geq \\ &\geq |T|/2 + \frac{1}{2} o\left(\{|X_{e_1}|, |X_{e_2} - X_{e_1}|, |X_{e_3} - X_{e_2} - X_{e_1}|, \cdots\}\right). \end{aligned} \tag{30}$$

It is easy to show the following estimations

$$C_k(G) = \frac{1}{2} \cdot \frac{n^{\underline{k}}}{k}; \tag{31}$$

$$C_k(G; e) = (n-2)^{\underline{k-2}} \text{ for } e \in E(G) \tag{32}$$

$$|T| = |C(G)| = \frac{1}{2} \sum_{3 \leq k \leq n} \frac{n^{\underline{k}}}{k} = \frac{1}{2}\left(\frac{n^{\underline{n}}}{n} + \frac{n^{\underline{n-1}}}{n-1} + \cdots + \frac{n^{\underline{3}}}{3}\right); \tag{33}$$

(34)
$$|X_e| = |C(G; e)| = \sum_{3 \leq k \leq n} (n-2)^{\underline{k-2}} = (n-2)^{\underline{n-2}} + (n-2)^{\underline{n-3}} + \cdots + (n-2)^{\underline{1}}.$$

We consider in G the edges incident in 1 and for $1 \leq t \leq n-1$ we have

$$\begin{aligned} &\left|X_{e_t} - X_{e_{t-1}} - \cdots - X_{e_1}\right| = |X_{e_t}| - \left|X_{e_t} \cap \left(X_{e_{t-1}} \cup \cdots \cup X_{e_1}\right)\right| = \\ &= |X_{e_t}| - \left|\left(X_{e_t} \cap X_{e_{t-1}}\right) \cup \cdots \cup \left(X_{e_t} \cap X_{e_1}\right)\right| = \\ &= |X_{e_t}| - \sum_{1 \leq i \leq t-1} |X_{e_t} \cap X_{e_i}| = \\ &= (n-2)\left[(n-3)^{\underline{n-3}} + (n-3)^{\underline{n-4}} + \cdots + (n-3)^{\underline{1}} + (n-3)^{\underline{0}}\right] - \\ &- (t-1)\left[(n-3)^{\underline{n-3}} + (n-3)^{\underline{n-4}} + \cdots + (n-3)^{\underline{1}} + 1\right] = \\ &= (n-t-1)\left[(n-3)^{\underline{n-3}} + (n-3)^{\underline{n-4}} + \cdots + (n-3)^{\underline{1}} + (n-3)^{\underline{0}}\right]. \end{aligned}$$

We obtain

$$\begin{aligned} &o\left(\left\{|X_{e_1}|, |X_{e_2} - X_{e_1}|, |X_{e_3} - X_{e_2} - X_{e_1}|, \cdots, \left|X_{e_{n-1}} - X_{e_{n-2}} - \cdots - X_{e_1}\right|\right\}\right) \\ &= \begin{cases} 0 & \text{for } n \equiv \ (\text{mod}2) \\ \dfrac{n-1}{2} & \text{for } n \equiv 1 \ (\text{mod}2). \end{cases} \end{aligned}$$

We continue the estimation for the graph $G - \{1\}$ and the edges incident in 2, for the graph $G - \{1, 2\}$ and the edges incident in 3, etc.

Finally we obtain

$$o\left(\{|X_{e_1}|, |X_{e_2} - X_{e_1}|, |X_{e_3} - X_{e_2} - X_{e_1}|, \cdots\}\right) = \tag{35}$$

$$= \begin{cases} \dfrac{n-2}{2} + \dfrac{n-4}{2} + \dfrac{n-6}{2} + \cdots + \dfrac{4}{2} + \dfrac{2}{2} \text{ for } n \equiv 0(\mathrm{mod}2) \\ \dfrac{n-1}{2} + \dfrac{n-3}{2} + \dfrac{n-5}{2} + \cdots + \dfrac{4}{2} + \dfrac{2}{2} \text{ for } n \equiv 1(\mathrm{mod}2). \end{cases} =$$

$$= 1 + 2 + 3 + \cdots + \lfloor \frac{n-1}{2} \rfloor = \frac{1}{2} \cdot \lfloor \frac{n-1}{2} \rfloor \cdot \lfloor \frac{n+1}{2} \rfloor$$

and the Theorem follows from (30), (33) and (35). □

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