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On the Forcing Semantics for Monoidal t-norm Based Logic[1]

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Abstract: MTL-algebras are algebraic structures for the Esteva-Godo monoidal t-norm based logic (MTL), a many-valued propositional calculus that formalizes the structure of the real interval $[0, 1]$, induced by a left-continuous t-norm. Given a complete MTL-algebra $\mathcal{X}$, we define the weak forcing value $|\varphi|_{\mathcal{X}}$ and the forcing value $[\varphi]_{\mathcal{X}}$, for any formula φ of MTL in $\mathcal{X}$. We establish some arithmetical properties of $|.|_{\mathcal{X}}$ and $[.]_{\mathcal{X}}$, and prove the equality $[\varphi]_{\mathcal{X}}=\|\varphi\|_{\mathcal{X}}$, where $\|\varphi\|_{\mathcal{X}}$ is the truth value of φ in $\mathcal{X}$.

Key Words: MTL logic, MTL-algebras, forcing semantics
Category: F.4.1

1 Introduction

In many cases, the approximate reasoning operates with a conjunction which generalize the one in the classical logic. The triangular norm (t-norm) is a good candidate for modelling this kind of conjunction [Bělohlavek 2002, Gottwald 2005, Klement et.al. 2000].

The structure defined by a continuous t-norm on the interval $[0, 1]$ constitutes the base for Hajek's Basic Logic (BL) [Hájek 1998a, Hájek 1998b] and for BL-algebras, the structures canonically associated to BL [Hájek and Ševčik 2004, Cintula and Hájek 2006].

More generally, the Esteva-Godo logic MTL and MTL-algebras correspond to the left-continuous t-norms and their residua [Esteva and Godo 2001]. The completeness theorems for MTL (and for the derived logical systems) concerns with the usual algebraic semantic [Esteva et.al. 2002]. Another kind of semantics for MTL (named Kripke semantics) are discussed in [Montagna and Ono 2002,

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Montagna and Sacchetti 2004]. The Kripke semantics for MTL are based on the notion of r-forcing.

The concept of truth value is the usual way to evaluate the formulas of MTL. For a formula φ of MTL, the truth value $\|\varphi\|_{\mathcal{X}}$ of φ is defined in an MTL-algebra $\mathcal{X}$.

In this paper we shall adopt an alternative point of view: for any formula φ of MTL and for any complete MTL-algebra $\mathcal{X}$, we define the weak forcing value $|\varphi|_{\mathcal{X}}$ and the forcing value $[\varphi]_{\mathcal{X}}$ of φ in $\mathcal{X}$. These two semantics correspond to the notions of forcing and r-forcing studied in [Montagna and Ono 2002, Montagna and Sacchetti 2004]. Thus, instead of talking about "the formula φ is valid in a Kripke model", we calculate $|\varphi|_{\mathcal{X}}$ or $[\varphi]_{\mathcal{X}}$.

Section 2 contains some basic notions and results on residuated lattices and MTL-algebras. Some elements of syntax and semantic of MTL are recalled in Section 3.

In Section 4 we establish a lot of properties regarding the behaviour of the weak forcing value w.r.t. some types of formulas of MTL. In Section 5 we continue to study the behaviour of $|\cdot|_{\mathcal{X}}$ w.r.t. some formulas of MTL (especially the axioms of MTL) and compare the truth value semantics with the weak forcing semantic.

The main result of this paper (Theorem 19) shows that $[\varphi]_{\mathcal{X}} = \|\varphi\|_{\mathcal{X}}$, for any formula φ of MTL and for any complete MTL-algebra $\mathcal{X}$. The equality $[.]_{\mathcal{X}} = \|.\|_{\mathcal{X}}$ improves the relationship between Kripke-style semantic and algebraic semantic studied in [Montagna and Ono 2002, Montagna and Sacchetti 2004].

Section 7 contains some suggestions for further research on $|.|_{\mathcal{X}}$ and $[.]_{\mathcal{X}}$ in the framework of predicate logic $MTL\forall$ and of some non-commutative fuzzy logics associated to MTL and $MTL\forall$.

2 MTL-algebras

A *residuated lattice* is a structure $\mathcal{A}= (A, \vee, \wedge, \cdot, \rightarrow, 0, 1)$ equipped with an order $\leq$ satisfying the following:

i) $(A, \vee, \wedge, 0, 1)$ is a bounded lattice;

ii) $(A, \cdot, 1)$ is a commutative monoid;

iii) For any $a, b, c \in A$, $a \cdot b \leq c$ iff $a \leq b \rightarrow c$.

We shall write ab instead of $a \cdot b$.

In a residuated lattice $\mathcal{A}$, *the negation* $^-$ is introduced by $\overline{a} = a \rightarrow 0$, for any $a \in A$.

Lemma 1. *[Bělohlavek 2002] Let $\mathcal{A}$ be a residuated lattice. Then, for all $a, b, c \in A$, the following hold:*

(1) $a \leq b$ *iff* $a \rightarrow b = 1$*;*

(2) $a \cdot 0 = 0$*;*

(3) $1 \rightarrow a = a$*;*

(4) $ab \leq a$*;*

(5) $a(a \rightarrow b) \leq b$*;*

(6) $a \rightarrow (b \rightarrow c) = b \rightarrow (a \rightarrow c) = ab \rightarrow c$*;*

(7) If $b \leq c$*, then* $a \rightarrow b \leq a \rightarrow c$ *and* $c \rightarrow a \leq b \rightarrow a$*;*

(8) If $a \leq b$*, then* $ac \leq bc$*;*

(9) $a \rightarrow a = 1$.

Lemma 2. *[Bělohlavek 2002] Let* $\mathcal{A}$ *be a residuated lattice. Then, for all elements* $a \in A$ *and* $\{a_i\}_{i \in I} \subseteq A$*, the following hold:*

(1) $(\bigvee_{i \in I} a_i) \rightarrow a = \bigwedge_{i \in I}(a_i \rightarrow a)$*;*

(2) $a \rightarrow (\bigwedge_{i \in I} a_i) = \bigwedge_{i \in I}(a \rightarrow a_i)$*;*

(3) $a(\bigvee_{i \in I} a_i) = \bigvee_{i \in I} aa_i$*;*

(4) $\bigvee_{i \in I}(a \rightarrow a_i) \leq a \rightarrow (\bigvee_{i \in I} a_i)$*;*

(5) $\bigvee_{i \in I}(a_i \rightarrow a) \leq (\bigwedge_{i \in I} a_i) \rightarrow a$.

An MTL*-algebra* [Esteva and Godo 2001] is a residuated lattice $\mathcal{A}$ such that, for all $a, b \in A$, we have

(iv) $(a \rightarrow b) \vee (b \rightarrow a) = 1$.

Example. A t*-norm* is a binary operation $*$ on the interval $[0, 1]$ which is associative, commutative, non-decreasing in the both arguments and the identity $a * 1 = a$ holds. If $*$ is a left-continuous t-norm, then $([0, 1], \vee, \wedge, *, \rightarrow, 0, 1)$ is an MTL-algebra, where the residuum operation $\rightarrow$ on $[0, 1]$ is defined by

$$a \rightarrow b = \bigvee \{c \mid a * c \leq b\}.$$

This structure will be called a *standard* MTL*-algebra*.

Any totally-ordered residuated lattice $\mathcal{A}$ is an MTL-algebra. In this case, $\mathcal{A}$ will be called an MTL*-chain*. By [Cintula and Hájek 2006], any MTL-algebra is isomorphic to a subdirect product of MTL-chains.

Lemma 3. *([Bělohlavek 2002], Theorem 2.34) If $\mathcal{A}$ is a residuated lattice, then the following conditions are equivalent:*

(i) *$\mathcal{A}$ is an MTL-algebra;*

(ii) *For all $a, b, c \in A$, $a \to (b \vee c) = (a \to b) \vee (a \to c)$;*

(iii) *For all $a, b, c \in A$, $(b \wedge c) \to a = (b \to a) \vee (c \to a)$.*

3 Monoidal *t*-norm based logic

In this section we shall recall some basic notions of the monoidal t-norm based logic (MTL) (see [Esteva and Godo 2001, Esteva et.al. 2002]).

The language of MTL has the following primitive symbols:

- denumerable many propositional variables (V will denote the set of propositional variables);
- the connectives $\vee, \wedge, \odot, \to$;
- the symbol $\perp$;
- the parenthesis (,).

The set $Form$ of formulas of MTL is defined as usual. Let us denote $\top = 1 \to \perp$. We list the axioms of MTL:

(A1) $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$;

(A2) $\varphi \odot \psi \to \psi$;

(A3) $\varphi \odot \psi \to \psi \odot \varphi$;

(A4) $\varphi \wedge \psi \to \varphi$;

(A5) $\varphi \wedge \psi \to \psi \wedge \varphi$;

(A6) $\varphi \odot (\varphi \to \psi) \to (\varphi \wedge \psi)$;

(A7) $(\varphi \to (\psi \to \chi)) \to ((\varphi \odot \psi) \to \chi)$;

(A8) $((\varphi \odot \psi) \to \chi) \to (\varphi \to (\psi \to \chi))$;

(A9) $(\varphi \to (\psi \to \chi)) \to (((\psi \to \varphi) \to \chi) \to \chi)$;

(A10) $\perp \to \varphi$.

Modus-ponens is the only rule of inference of MTL: $\frac{\varphi, \varphi\to\psi}{\psi}$.

The notion of provable formula is defined as usual. We denote by $\vdash \varphi$ that the formula φ is provable in MTL.

Let Σ be a subset of the set of axioms (A1)-(A10). If φ is a formula of MTL, then we denote by $\vdash_\Sigma \varphi$ that φ can be derived from Σ by using modus-ponens; if Σ is the set of all axioms (A1)-(A10), then $\vdash_\Sigma \varphi$ means that $\vdash \varphi$.

Let $\mathcal{X}= (X, \vee, \wedge, \cdot, \to, 0, 1)$ be an MTL-algebra. An *evaluation* of MTL in $\mathcal{X}$ is a function $e : V \to X$. Any evaluation $e : V \to X$ can be uniquely extended to a function $\hat{e} : Form \to X$ with the property that for all $\varphi, \psi \in Form$ we have:

(a) $\hat{e}(\varphi) = e(\varphi)$, if $\varphi \in V$;

(b) $\hat{e}(\perp) = 0$;

(c) $\hat{e}(\varphi \vee \psi) = \hat{e}(\varphi) \vee \hat{e}(\psi)$;

(d) $\hat{e}(\varphi \wedge \psi) = \hat{e}(\varphi) \wedge \hat{e}(\psi)$;

(e) $\hat{e}(\varphi \odot \psi) = \hat{e}(\varphi) \cdot \hat{e}(\psi)$;

(f) $\hat{e}(\varphi \to \psi) = \hat{e}(\varphi) \to \hat{e}(\psi)$.

The *truth value* $\|\varphi\|_\mathcal{X}$ of a formula φ in $\mathcal{X}$ is defined by:

$$\|\varphi\|_\mathcal{X} = \bigwedge\{\hat{e}(\varphi) \mid e \text{ is an evaluation in } \mathcal{X}\}.$$

4 Weak forcing value of a formula of MTL

In this section we shall define the weak forcing value $|\varphi|_\mathcal{X}$ of a formula φ of MTL in a complete MTL-algebra $\mathcal{X}$. Besides the truth value $\|\varphi\|_\mathcal{X}$ of φ in $\mathcal{X}$, $|\varphi|_\mathcal{X}$ constitutes an alternative to evaluate the formula φ in $\mathcal{X}$. The weak forcing value is a rafinement of the notion of validity in a Kripke model (in the sense of [Montagna and Ono 2002, Montagna and Sacchetti 2004]).

We fix a complete MTL-algebra $\mathcal{X}= (X, \vee, \wedge, \cdot, \to, 0, 1)$.

Definition 4. An **$\mathcal{X}$-valued weak forcing property** is a function

$$f : (V \cup \{\perp\}) \times X \to X$$

such that the following conditions hold:

(i) If $\varphi \in V$ and $x, y \in X$, then $x \le y$ implies $f(\varphi, y) \le f(\varphi, x)$;

(ii) $f(\perp, 1) = 0$.

Definition 5. Let f be an $\mathcal{X}$-valued weak forcing property. For any $\varphi \in Form$ and $x \in X$, we define, by induction, the element $[\varphi]^f_x$ of X :

(1) $[\varphi]^f_x = f(\varphi, x)$, if $\varphi \in V$;

(2) $[\bot]^f_x = \overline{x}$;

(3) If $\varphi = \alpha \vee \beta$, then $[\varphi]^f_x = [\alpha]^f_x \vee [\beta]^f_x$;

(4) If $\varphi = \alpha \wedge \beta$, then $[\varphi]^f_x = [\alpha]^f_x \wedge [\beta]^f_x$;

(5) If $\varphi = \alpha \odot \beta$, then $[\varphi]^f_x = \bigvee_{y,z \in X}((x \to yz)[\alpha]^f_y \ [\beta]^f_z)$;

(6) If $\varphi = \alpha \to \beta$, then $[\varphi]^f_x = \bigwedge_{y \in X}([\alpha]^f_y \to [\beta]^f_{xy})$.

For simplicity, we shall usually write $[\varphi]_x$ instead of $[\varphi]^f_x$.

Definition 6. The **weak forcing value** $|\varphi|_{\mathcal{X}}$ of a formula φ in $\mathcal{X}$ is defined by

$$|\varphi|_{\mathcal{X}} = \bigwedge\{[\varphi]^f_1 \mid \text{f is an } \mathcal{X}\text{-valued weak forcing property }\}.$$

Lemma 7. *Let f be an $\mathcal{X}$-valued weak forcing property. For any formula φ of MTL and $y \leq x$ in X, we have $[\varphi]_x \leq [\varphi]_y$.*

Proof. We proceed by induction on the complexity of φ. We treat only the case $\varphi = \alpha \to \beta$. If $y \leq x$, then $yz \leq xz$, hence, by induction hypothesis, we have $[\beta]_{xz} \leq [\beta]_{yz}$, for all $z \in X$. Then, by Lemma 1, (7), we get

$$[\varphi]_x = \bigwedge_{z \in X}(\ [\alpha]_z \to [\beta]_{xz}\ ) \leq \bigwedge_{z \in X}(\ [\alpha]_z \to [\beta]_{yz}\ ) = [\varphi]_y\ .$$

Remark. By Lemma 7, $[\varphi]_1 \leq [\varphi]_x$, for any $x \in X$.

In what follows, we emphasize the behaviour of $[\cdot]^f$ and $|\cdot|_{\mathcal{X}}$ w.r.t. some formulas of MTL.

Proposition 8. *Let f be an $\mathcal{X}$-valued weak forcing property. For all formulas φ, ψ, χ of MTL and $x, y, a, b, c, p, q, t \in X$, the following hold:*

(1) $[\varphi \to \varphi]_x = 1$*;*

(2) $[\top]_x = 1$*;*

(3) $[\psi]_x \leq [\varphi \to \psi]_x$*;*

(4) $[\varphi]_x \cdot [\varphi \to \psi]_y \leq [\psi]_{xy}$*;*

(5) $[\varphi]_x \cdot [\varphi \to \psi]_x \leq [\psi]_{x^2}$*;*

(6) $[\varphi \to \psi]_a \cdot [\psi \to \chi]_b \leq [\varphi]_c \to [\chi]_{abc}$*;*

(7) $[\varphi \to \psi]_x \leq [\psi \to \chi]_y \to [\varphi \to \chi]_{xy}$*;*

(8) $[\varphi \to (\psi \to \chi)]_x = \bigwedge_{u,v \in X}(\ [\varphi]_u\ [\psi]_v \to [\chi]_{xuv})$*;*

(9) $[\varphi \odot \psi \to \chi]_x = \bigwedge_{p,q,t \in X}((t \to pq)\ [\varphi]_p\ [\psi]_q \to [\chi]_{tx})$;

(10) $[\varphi \to (\psi \to \chi)]_x = [\psi \to (\varphi \to \chi)]_x$;

(11) $[(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))]_x = 1$;

(12) $[(\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi))]_x = 1$;

(13) $[\varphi \odot \psi \to \chi)]_x \leq [\varphi \to (\psi \to \chi)]_x$;

(14) $[(\varphi \odot \psi \to \chi) \to (\varphi \to (\psi \to \chi))]_x = 1$;

(15) $[(\varphi \to \psi) \wedge (\varphi \to \chi)]_x = [\varphi \to (\psi \wedge \chi)]_x$;

(16) $[\varphi \odot (\psi \vee \chi)]_x = [(\varphi \odot \psi) \vee (\varphi \odot \chi)]_x$.

Proof.
(1) By Lemma 7, $[\varphi \to \varphi]_x \geq [\varphi \to \varphi]_1 = \bigwedge_{u \in X}([\varphi]_u \to [\varphi]_u) = 1$.

(2) Since $\top$ is $\bot \to \bot$, by (1) we obtain $[\top]_x = 1$.

(3) By Lemma 1, (4), and Lemma 7, $[\psi]_x \leq [\psi]_{ux}$, for each $u \in X$. Then, by Lemma 1, (1), (7), $[\psi]_x \leq [\varphi]_u \to [\psi]_x \leq [\varphi]_u \to [\psi]_{ux}$ for each $u \in X$. Hence $[\psi]_x \leq \bigwedge_{u \in X}([\varphi]_u \to [\psi]_{ux}) = [\varphi \to \psi]_x$.

(4) According to Lemma 1, (5), we have

$$[\varphi]_x \cdot [\varphi \to \psi]_y = [\varphi]_x \cdot \bigwedge_{u \in X}([\varphi]_u \to [\psi]_{yu}) \leq [\varphi]_x \cdot ([\varphi]_x \to [\psi]_{xy}) \leq [\psi]_{xy}.$$

(5) By *(4)*.

(6) Using *(4)*, we have $[\varphi]_c \cdot [\varphi \to \psi]_a \cdot [\psi \to \chi]_b \leq [\psi]_{ac} \cdot [\psi \to \chi]_b \leq [\chi]_{abc}$, so, the inequality $[\varphi \to \psi]_a \cdot [\psi \to \chi]_b \leq [\varphi]_c \to [\chi]_{abc}$ follows.

(7) According to *(6)*, for each $u \in X$ we have $[\varphi \to \psi]_x \cdot [\psi \to \chi]_y \leq [\varphi]_u \to [\chi]_{uxy}$, so $[\varphi \to \psi]_x \cdot [\psi \to \chi]_y \leq \bigwedge_{u \in X}([\varphi]_u \to [\chi]_{uxy}) = [\varphi \to \chi]_{xy}$. Hence $[\varphi \to \psi]_x \leq [\psi \to \chi]_y \to [\varphi \to \chi]_{xy}$.

(8) Applying the clause (6) of Definition 5, Lemma 2, (2), and Lemma 1, (6), we obtain

$$\begin{aligned}[\varphi \to (\psi \to \chi)]_x &= \bigwedge_{u \in X}([\varphi]_u \to [\psi \to \chi]_{xu}) = \\ &= \bigwedge_{u \in X}([\varphi]_u \to \bigwedge_{v \in X}([\psi]_v \to [\chi]_{xuv})) = \\ &= \bigwedge_{u,v \in X}(\ [\varphi]_u\ [\psi]_v \to [\chi]_{xuv}).\end{aligned}$$

(9) We apply the clauses (6) and (5) of Definition 5 and Lemma 2, (1), and we obtain

$$\begin{aligned}[\varphi \odot \psi \to \chi]_x &= \bigwedge_{t \in X}([\varphi \odot \psi]_t \to [\chi]_{tx}) = \\ &= \bigwedge_{t \in X}((\bigvee_{p,q \in X}(t \to pq)[\varphi]_p\ [\psi]_q) \to [\chi]_{tx}) =\end{aligned}$$

$= \bigwedge_{p,q,t\in X}((t \to pq)[\varphi]_p\ [\psi]_q \to [\chi]_{tx}))$.

(10) By *(8)*.

(11) By *(8)*, Lemma 7 and *(6)* it follows that

$$\begin{aligned}
&[(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))]_x = \\
&= \bigwedge_{u,v\in X}(\ [\varphi \to \psi]_u\ [\psi \to \chi]_v \to \bigwedge_{w\in X}(\ [\varphi]_w \to [\chi]_{xuvw})) = \\
&= \bigwedge_{u,v,w\in X}(\ [\varphi]_w\ [\varphi \to \psi]_u\ [\psi \to \chi]_v \to [\chi]_{xuvw}) \geq \\
&\geq \bigwedge_{u,v,w\in X}(\ [\varphi]_w\ [\varphi \to \psi]_u\ [\psi \to \chi]_v \to [\chi]_{uvw}) = 1.
\end{aligned}$$

(12) Applying Lemma 7 and *(10)*, we get

$$\begin{aligned}
&[(\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi))]_x = \\
&= \bigwedge_{u\in X}(\ [\varphi \to (\psi \to \chi)]_u \to [\psi \to (\varphi \to \chi)]_{ux}) \geq \\
&\geq \bigwedge_{u\in X}(\ [\varphi \to (\psi \to \chi)]_u \to [\psi \to (\varphi \to \chi]_u) = 1.
\end{aligned}$$

(13) Let $u, v \in X$. By *(9)*, $[\varphi \odot \psi \to \chi]_x \leq [\varphi]_u\ [\psi]_v \to [\chi]_{xuv}$, hence, by *(8)*, we get $[\varphi \odot \psi \to \chi]_x \leq \bigwedge_{u,v\in X}(\ [\varphi]_u\ [\psi]_v \to [\chi]_{xuv}) = [\varphi \to (\psi \to \chi)]_x$.

(14) Similar to *(12)*.

(15) We have the following

$$\begin{aligned}
&[(\varphi \to \psi) \wedge (\varphi \to \chi)]_x = [\varphi \to \psi]_x \wedge [\varphi \to \chi]_x = \\
&= \bigwedge_{y\in X}([\varphi]_y \to [\psi]_{xy}) \wedge \bigwedge_{y\in X}([\varphi]_y \to [\chi]_{xy}) = \\
&= \bigwedge_{y\in X}(([\varphi]_y \to [\psi]_{xy}) \wedge ([\varphi]_y \to [\chi]_{xy})) = \\
&= \bigwedge_{y\in X}([\varphi]_y \to ([\psi]_{xy} \wedge [\chi]_{xy})) = \bigwedge_{y\in X}([\varphi]_y \to [\psi \wedge \chi]_{xy}) = \\
&= [\varphi \to (\psi \wedge \chi)]_x.
\end{aligned}$$

(16) We can write

$$\begin{aligned}
&[(\varphi \odot \psi) \vee (\varphi \odot \chi)]_x = [\varphi \odot \psi]_x \vee [\varphi \odot \chi]_x = \\
&= (\bigvee_{y,z\in X}(x \to yz)[\varphi]_y[\psi]_z) \vee (\bigvee_{y,z\in X}(x \to yz)[\varphi]_y[\chi]_z) = \\
&= \bigvee_{y,z\in X}(((x \to yz)[\varphi]_y[\psi]_z) \vee ((x \to yz)[\varphi]_y[\chi]_z)) = \\
&= \bigvee_{y\in X}(x \to yz)[\varphi]_y([\psi]_z \vee [\chi]_z) = \\
&= \bigvee_{y,z\in X}(x \to yz)\ [\varphi]_y\ [\psi \vee \chi]_z = [\varphi \odot (\psi \vee \chi)]_x.
\end{aligned}$$

Corollary 9. *For any formulas φ, ψ and χ of MTL, the following hold:*

(1) $|\varphi \to \varphi|_{\mathcal{X}} = 1$;

(2) $|\top|_{\mathcal{X}} = 1$;

(3) $|\psi|_{\mathcal{X}} \leq |\varphi \to \psi|_{\mathcal{X}}$;

(4) $|\varphi \to (\psi \to \chi)|_{\mathcal{X}} = |\psi \to (\varphi \to \chi)|_{\mathcal{X}}$;

(5) $|(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))|_{\mathcal{X}} = 1$;

(6) $|(\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi))|_{\mathcal{X}} = 1$;

(7) $|(\varphi \odot \psi \to \chi) \to (\varphi \to (\psi \to \chi))|_{\mathcal{X}} = 1$.

Corollary 10. *If* $|\varphi|_{\mathcal{X}} = |\varphi \to \psi|_{\mathcal{X}} = 1$, *then* $|\psi|_{\mathcal{X}} = 1$.

Remark. Assume that Σ is the set of axioms (A1), (A3), (A4), (A8), (A10) and φ is a formula of MTL. By Corollaries 9 and 10, if $\vdash_{\Sigma} \varphi$, then $|\varphi|_{\mathcal{X}} = 1$.

Proposition 11. *Let f be an $\mathcal{X}$-valued weak forcing property. For any formula φ of MTL and $x, y \in X$, we have:*

(1) $[\neg\varphi]_x = \bigwedge_{y\in X}(xy[\varphi]_y)^- = x \to \bigwedge_{y\in X}(y[\varphi]_y)^-$;

(2) $xy[\neg\varphi]_x[\varphi]_y = 0$;

(3) $[\varphi]_x \leq [\neg\neg\varphi]_x$;

(4) $[\neg\varphi]_x = [\neg\neg\neg\varphi]_x$;

(5) $[\neg(\varphi \vee \psi)]_x = [\neg\varphi \wedge \neg\psi]_x$;

(6) $[\varphi \to \psi]_x \leq [\neg\psi \to \neg\varphi]_x$;

(7) $[\varphi \to \neg\psi]_x \leq [\psi \to \neg\varphi]_x$;

(8) $x[\varphi \odot \neg\varphi]_x = 0$;

(9) $[\varphi \to (\psi \odot \neg\psi)]_x \leq [\neg\varphi]_x$.

Proof.
(1) $[\neg\varphi]_x = \bigwedge_{y\in X}([\varphi]_y \to [\bot]_{xy}) = \bigwedge_{y\in X}([\varphi]_y \to \overline{xy}) = \bigwedge_{y\in X}(xy[\varphi]_y)^-$.
In a similar way we get $[\neg\varphi]_x = x \to \bigwedge_{y\in X}(y[\varphi]_y)^-$.

(2) By *(1)*, for any $y \in X$, we have $[\neg\varphi]_x \leq (xy[\varphi]_y)^-$, hence $xy[\neg\varphi]_x[\varphi]_y = 0$.

(3) Let $y \in X$. By *(2)*, $xy[\varphi]_x[\neg\varphi]_y = 0$, hence $x[\varphi]_x \leq (y[\neg\varphi]_y)^-$.
Thus $x[\varphi]_x \leq \bigwedge_{y\in X}(y[\neg\varphi]_y)^-$, so $[\varphi]_x \leq x \to \bigwedge_{y\in X}(y[\neg\varphi]_y)^- = [\neg\neg\varphi]_x$.

(4) Let $y \in X$. By *(3)* and *(2)* we get $xy[\varphi]_y[\neg\neg\neg\varphi]_x \leq xy[\neg\neg\varphi]_y[\neg\neg\neg\varphi]_x = 0$, therefore $x[\neg\neg\neg\varphi]_x \leq (y[\varphi]_y)^-$. Thus $x[\neg\neg\neg\varphi]_x \leq \bigwedge_{y\in X}(y[\varphi]_y)^-$, hence $[\neg\neg\neg\varphi]_x \leq x \to \bigwedge_{y\in X}(y[\varphi]_y)^- = [\neg\varphi]_x$. The converse implication follows by *(3)*.

(5) We have

$$\begin{aligned}[\neg\varphi \wedge \neg\psi]_x &= [\neg\varphi]_x \wedge [\neg\psi]_x = \bigwedge_{y\in X}(xy[\varphi]_y)^- \wedge \bigwedge_{y\in X}(xy[\psi]_y)^- = \\ &= \bigwedge_{y\in X}((xy[\varphi]_y)^- \wedge \bigwedge_{y\in X}(xy[\psi]_y)^-) = \\ &= \bigwedge_{y\in X}(xy[\varphi]_y \vee xy[\psi]_y)^- = \bigwedge_{y\in X}(xy([\varphi]_y \vee [\psi]_y)^-) = \\ &= \bigwedge_{y\in X}(xy[\varphi \vee \psi]_y)^- = [\neg(\varphi \wedge \psi)]_x.\end{aligned}$$

(6) Let $y, z \in X$. According to Proposition 8, (4), $[\varphi]_y[\varphi \to \psi]_x \leq [\psi]_{xy}$, hence, by *(3)*, we get $[\varphi \to \psi]_x \cdot [\neg\psi]_z \cdot xyz \cdot [\varphi]_y \leq xyz \cdot [\psi]_{xy}[\neg\psi]_z = 0$. Then, for each $y \in X$, we have $[\varphi \to \psi]_x[\neg\psi]_z \leq (xyz[\varphi]_y)^-$, therefore $[\varphi \to \psi]_x[\neg\psi]_z \leq \bigwedge_{y\in X}(xyz[\varphi]_y)^- = [\neg\varphi]_{xz}$.

It follows that $[\varphi \to \psi]_x \leq [\neg\psi]_z \to [\neg\varphi]_{xz}$. This last inequality holds for each $z \in X$, therefore $[\varphi \to \psi]_x \leq \bigwedge_{z\in X}([\neg\psi]_z \to [\neg\varphi]_{xz}) = [\neg\psi \to \neg\varphi]_x$.

(7) Similar to *(6)*.

(8) According to Definition 5, (5), and the previous equality *(2)*, one obtains $x[\varphi \odot \neg\varphi]_x = x\bigvee_{y,z\in X}(x \to yz)[\varphi]_y[\neg\varphi]_z = \bigvee_{y,z\in X} x(x \to yz)[\varphi]_y[\neg\varphi]_z \leq \bigvee_{y,z\in X} yz[\varphi]_y[\neg\varphi]_z = 0$.

(9) Let $y \in X$. By Proposition 8, (4), and the previous equality *(8)*, we get $[\varphi \to (\psi \odot \neg\psi)]_x \cdot xy[\varphi]_y \leq xy[\psi \odot \neg\psi]_{xy} = 0$. Thus $[\varphi \to (\psi \odot \neg\psi)]_x \leq (xy[\varphi]_y)^-$, for any $y \in X$, hence $[\varphi \to (\psi \odot \neg\psi)]_x \leq \bigwedge_{y\in X}(xy[\varphi]_y)^- = [\neg\varphi]_x$.

5 The behaviour of $|\cdot|_{\mathcal{X}}$ w.r.t. some formulas of MTL

In this section we will compare the two kinds of semantics: truth value and weak forcing. A formula φ of MTL is valid in the weak forcing semantic iff $[\varphi]_1^f = 1$, for any $\mathcal{X}$-valued weak forcing property.

In the following we will analyze the behaviour of $|\cdot|_{\mathcal{X}}$ w.r.t. the axioms of MTL and some other formulas. We will prove that some axioms are valid via the new kind of semantics, while others are not valid (in this latter case we will provide a counterexample of a weak forcing property f for which $[\varphi]_1^f \neq 1$).

This analysis is very important in providing the similarities and the differences between the two semantics $|.|_{\mathcal{X}}$ and $\|.\|_{\mathcal{X}}$.

(A1) $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$

By Corollary 9, (5), this axiom is valid w.r.t. the weak forcing semantic.

(A2) $\varphi \odot \psi \to \psi$

Let us consider $\mathcal{X}= L_3 = \{0, \frac{1}{2}, 1\}$ with the canonical structure of MV_3-algebra [2].

[2] An MV-*algebra* is a structure $(A, \oplus, \odot, ^-, 0, 1)$, where $\oplus$ and $\odot$ are binary operations, $^-$ is unary and 0,1 are constants, satisfying the following axioms:

a) $(A, \oplus, 0)$ and $(A, \odot, 1)$ are commutative monoids,

b) $x \odot 0 = 0$ and $x \oplus 1 = 1$, for any $x \in A$,

c) $x^{--} = x$, for any $x \in A$,

Let us consider $p, q \in V$ (some propositional variables) and the weak forcing property f which has the following behaviour w.r.t. p and q:

f	0	$\frac{1}{2}$	1
p	1	1	1
q	1	1	0

We have the followings:

$[p \odot q]_0^f = \bigvee_{y,z \in L_3} \cdot f(p, y) \cdot f(q, z) = 1$

$[p \odot q]_{\frac{1}{2}}^f = \bigvee_{y,z \in L_3} (\frac{1}{2} \to yz) \cdot f(p, y) \cdot f(q, z) = 1$

$[p \odot q]_1^f = \bigvee_{y,z \in L_3} (1 \to yz) \cdot f(p, y) \cdot f(q, z) = \frac{1}{2}$

$[p \odot q \to q]_1^f = \bigwedge_{x \in L_3} ([p \odot q]_x^f \to f(q, x)) = (1 \to 1) \wedge (1 \to 1) \wedge (\frac{1}{2} \to 0) = \frac{1}{2}$

Thus $[p \odot q \to q]_1^f = \frac{1}{2} \neq 1$ which prove that (A2) is not valid w.r.t. the new semantics.

(A3) $\varphi \odot \psi \to \psi \odot \varphi$

Let f be a weak forcing property. Using Lemma 1, (9), we get:

$[\varphi \odot \psi \to \psi \odot \varphi]_1^f = \bigwedge_{t \in L_3} ([\varphi \odot \psi]_t^f \to [\psi \odot \varphi]_t^f) =$

$= \bigwedge_{t \in L_3} [(\bigvee_{y,z \in L_3} (t \to yz) [\varphi]_y^f [\psi]_z^f) \to (\bigvee_{y,z \in L_3} (t \to yz) [\varphi]_y^f [\psi]_z^f)] = 1.$

Hence, the axiom is valid w.r.t. the weak forcing semantic.

(A4) $\varphi \wedge \psi \to \varphi$

Let f be a weak forcing property.

By Lemma 3, (3), and Lemma 1, (9), we have:

$[\varphi \wedge \psi \to \varphi]_1^f = \bigwedge_{y \in L_3} ([\varphi \wedge \psi]_y^f \to [\varphi]_y^f) = \bigwedge_{y \in L_3} (([\varphi]_y^f \wedge [\psi]_y^f) \to [\varphi]_y^f) =$

$= \bigwedge_{y \in L_3} (([\varphi]_y^f \to [\varphi]_y^f) \vee ([\psi]_y^f \to [\varphi]_y^f)) =$

$= \bigwedge_{y \in L_3} (1 \vee ([\psi]_y^f \to [\varphi]_y^f)) = 1.$

Therefore this axiom is valid in the new semantics.

---

d) $(x \oplus y)^- = x^- \odot y^-$, for any $x, y \in A$,

e) $(x \odot y^-) \oplus y = (y \odot x^-) \oplus x$, for any $x, y \in A$.

An MV_3-*algebra* is an MV-algebra with the property $x \oplus x \oplus x = x \oplus x$.

Any MV-algebra is an MTL-algebra, where the implication is given by $x \to y = \overline{x} \oplus y$.

(A5) $\varphi \wedge \psi \to \psi \wedge \varphi$

Let f be a weak forcing property. Using Lemma 3, (3), Lemma 2, (2), and the definition of an MTL-algebra, we obtain:

$$[\varphi \wedge \psi \to \psi \wedge \varphi]_1^f = \bigwedge_{y \in L_3}([\varphi \wedge \psi]_y^f \to [\psi \wedge \varphi]_y^f) =$$
$$= \bigwedge_{y \in L_3}(([\varphi]_y^f \wedge [\psi]_y^f) \to ([\psi]_y^f \wedge [\psi]_y^f)) =$$
$$= \bigwedge_{y \in L_3}[([\varphi]_y^f \to ([\psi]_y^f \wedge [\psi]_y^f)) \vee ([\psi]_y^f \to ([\psi]_y^f \wedge [\psi]_y^f))] =$$
$$= \bigwedge_{y \in L_3}[(([\varphi]_y^f \to [\psi]_y^f) \wedge ([\varphi]_y^f \to [\varphi]_y^f)) \vee (([\psi]_y^f \to [\psi]_y^f) \wedge ([\psi]_y^f \to [\varphi]_y^f))] =$$
$$= \bigwedge_{y \in X}[([\varphi]_y^f \to [\psi]_y^f) \vee ([\psi]_y^f \to [\varphi]_y^f)] = 1$$

Therefore this axiom is valid w.r.t. the weak forcing semantic.

(A6) $\varphi \odot (\varphi \to \psi) \to (\varphi \to \psi)$

Let us consider $\mathcal{X} = L_3 = \{0, \frac{1}{2}, 1\}$ with the canonical structure of MV_3-algebra and let $p, q \in V$ and f be a weak forcing property which has the following behaviour w.r.t. p, q:

f	0	$\frac{1}{2}$	1
p	1	1	1
q	1	1	0

Because $[p \to q]_x^f = \bigwedge_{y \in L_3} f(p, y) \to f(q, x \cdot y)$, we obtain $[p \to q]_0^f = 1$, $[p \to q]_{\frac{1}{2}}^f = 1$ and $[p \to q]_1^f = 0$.

We also have the followings:

$$[p \odot (p \to q)]_0^f = \bigvee_{t,z \in L_3} f(p, t) \cdot [p \to q]_z^f = 1$$

$$[p \odot (p \to q)]_{\frac{1}{2}}^f = \bigvee_{t,z \in L_3} (\tfrac{1}{2} \to tz) \cdot f(p, t) \cdot [p \to q]_z^f = 1$$

$$[p \odot (p \to q)]_1^f = \bigvee_{t,z \in L_3} (1 \to tz) \cdot f(p, t) \cdot [p \to q]_z^f = \tfrac{1}{2}$$

$$[p \odot (p \to q) \to (p \wedge q)]_1^f = \bigwedge_{x \in L_3}([p \odot (p \to q)]_x^f \to [p \wedge q]_x^f) =$$
$$= \bigwedge_{x \in L_3}([p \odot (p \to q)]_x^f \to (f(p, x) \wedge f(q, x))) =$$
$$= [1 \to (1 \wedge 1)] \wedge [1 \to (1 \wedge 1)] \wedge [\tfrac{1}{2} \to (1 \wedge 0)] = 1 \wedge 1 \wedge \tfrac{1}{2} = \tfrac{1}{2}$$

Thus, $[p \odot (p \to q) \to (p \wedge q)]_1^f = \frac{1}{2} \neq 1$. This prove that axiom (A6) is not valid w.r.t. the weak forcing semantic.

(A7) $(\varphi \to (\psi \to \chi)) \to ((\varphi \odot \psi) \to \chi)$

From [Iorgulescu 2004], the set $A = \{0, a, b, c, d, 1\}$ is organized as a lattice as in Figure 1 and as an MTL-algebra $\mathcal{A}$ with the operation $\to$ and $\odot$ as in the following tables:

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	1	1	1	1
b	a	a	1	1	1	1
c	0	a	d	1	d	1
d	a	a	c	c	1	1
1	0	a	b	c	d	1

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	0	0	a	0	a
b	0	0	b	b	b	b
c	0	a	b	c	b	c
d	0	0	b	b	d	d
1	0	a	b	c	d	1

Let us consider $p, q, r \in V$ and f a weak forcing property with the following behaviour w.r.t. p, q, r:

f	0	a	b	c	d	1
p	1	1	d	d	d	d
q	1	a	a	0	0	0
r	b	0	0	0	0	0

Because $[p \odot q]^f_x = \bigwedge_{y,z\in A}(x \to yz) \cdot f(p,y) \cdot f(q,z)$, we have $[p \odot q]^f_0 = 1$, $[p \odot q]^f_a = d$, $[p \odot q]^f_b = a$, $[p \odot q]^f_c = 0$, $[p \odot q]^f_d = a$ and $[p \odot q]^f_1 = 0$. We have the followings:

$$[(p \odot q) \to r]^f_1 = \bigwedge_{x\in A}([p \odot q]^f_x \to f(r, 1 \cdot x)) = \bigwedge_{x\in A}([p \odot q]^f_x \to f(r, x)) =$$
$$= (1 \to b) \wedge (d \to 0) \wedge (a \to 0) \wedge (0 \to 0) \wedge (a \to 0) \wedge (0 \to 0) = a$$

Because $[q \to r]^f_x = \bigwedge_{y\in A}(f(q,y) \to f(r, x \cdot y))$, we obtain that $[q \to r]^f_x = b$, for all $x \in A$. Then, we also have:

$$[p \to (q \to p)]^f_1 = \bigwedge_{x\in A}(f(p,x) \to [q \to r]^f_x) = (1 \to b) \wedge (d \to b) = b$$

Thus, $[p \to (q \to p)]^f_1 \to [(p \odot q) \to r]^f_1 = b \to a = a$.

By definition, we have

$$[(p \to (q \to r)) \to ((p \odot q) \to r)]^f_1 = \bigwedge_{x\in A}([p \to (q \to r)]^f_x \to [(p \odot q) \to r]^f_x),$$

therefore we have

$$[(p \to (q \to r)) \to ((p \odot q) \to r)]^f_1 \leq [p \to (q \to r)]^f_1 \to [(p \odot q) \to r]^f_1 = a.$$

Hence, axiom (A7) is not valid w.r.t. the new kind of semantics.

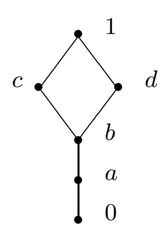

Figure 1:

(A8) $((\varphi \odot \psi) \to \chi) \to (\varphi \to (\psi \to \chi)$

By Corollary 9, (7), this axiom is valid w.r.t. the weak forcing semantic.

(A9) $(\varphi \to (\psi \to \chi)) \to (((\psi \to \varphi) \to \chi) \to \chi)$

Let us consider $\mathcal{X}= L_3 = \{0, \frac{1}{2}, 1\}$ with the canonical structure of MV_3-algebra. Let us also consider $p, q, r \in V$, some propositional variables, and f a weak forcing property with the following behaviour w.r.t. p, q, r:

f	0	$\frac{1}{2}$	1
p	0	0	0
q	$\frac{1}{2}$	0	0
r	0	0	0

Beacause $[q \to p]^f_x = \bigwedge_{y\in L_3}(f(q,y) \to f(p, x\cdot y))$, we obtain that $[q \to p]^f_0 = \frac{1}{2}$, $[q \to p]^f_{\frac{1}{2}} = \frac{1}{2}$ and $[q \to p]^f_1 = \frac{1}{2}$.

We have the followings:

$[p \to (q \to r)]^f_0 = \bigwedge_{z\in L_3}(f(p,z) \to [q \to r]^f_0) = \bigwedge_{z\in L3}(0 \to [q \to r]^f_0) = 1$

$[(q \to p) \to r]^f_x = \bigwedge_{y\in L3}([q \to p]^f_y \to f(r, x\cdot y)) = \bigwedge_{y\in L3}([q \to p]^f_y \to 0) = \frac{1}{2}$

$[((q \to p) \to r) \to r]^f_0 = \bigwedge_{x\in L_3}([(q \to p) \to r]^f_x \to f(r,0)) = \frac{1}{2}$

$[p \to (q \to r)]^f_0 \to [((q \to p) \to r) \to r]^f_0 = 1 \to \frac{1}{2} = \frac{1}{2}$

Because $[(p \to (q \to r)) \to (((q \to p) \to r) \to r)]^f_1 = \bigwedge_{x\in L_3}([p \to (q \to r)]^f_x \to [((q \to p) \to r) \to r]^f_x)$, we have that $[(p \to (q \to r)) \to (((q \to p) \to r) \to r)]^f_1 \le [p \to (q \to r)]^f_0 \to [((q \to p) \to r) \to r]^f_0 = \frac{1}{2}$, therefore axiom (A9) is not valid w.r.t. the weak forcing semantic.

(A10) $\perp \to \varphi$

Let us consider $\mathcal{X}= L_3 = \{0, \frac{1}{2}, 1\}$ with the canonical structure of MV_3-algebra and let $p \in V$ and f a weak forcing property such that $f(p,x) = 0$, for all $x \in L_3$. We have

$[\perp \to p]^f_1 = \bigwedge_{x\in L_3}([\perp]^f_x \to [p]^f_x) = \bigwedge_{x\in L_3}(\overline{x} \to f(p,x)) =$

$= (\overline{0} \to 0) \wedge (\overline{\frac{1}{2}} \to 0) \wedge (\overline{1} \to 0) = 0$

Therefore (A10) is not valid in the new semantics.

In the same way we can study the behaviour of $|\cdot|_{\mathcal{X}}$ w.r.t. some other formulas of MTL. For example, let us consider the formula $(\varphi \to \psi) \vee (\psi \to \varphi)$, where φ, ψ are MTL-formulas. From [Esteva and Godo 2001], we know that this formula is valid with respect to the truth value semantics. Now, let us consider L_3 with the canonical structure of MV_3-algebra and let $p, q \in V$. Let us consider a weak forcing property f with the following behaviour w.r.t. p, q:

f	0	$\frac{1}{2}$	1
p	1	1	0
q	1	$\frac{1}{2}$	$\frac{1}{2}$

By definition, we obtain:

$[p \to q]_1^f = \bigwedge_{y \in L_3}(f(p, y) \to f(q, y)) =$

$= (f(p, 0) \to f(q, 0)) \wedge (f(p, \frac{1}{2}) \to f(q, \frac{1}{2})) \wedge (f(p, 1) \to f(q, 1)) =$

$= (1 \to 1) \wedge (1 \to \frac{1}{2}) \wedge (0 \to \frac{1}{2}) = \frac{1}{2}$

$[q \to p]_1^f = \bigwedge_{y \in L_3}(f(q, y) \to f(p, y)) = (1 \to 1) \wedge (\frac{1}{2} \to 1) \wedge (\frac{1}{2} \to 0) = \frac{1}{2}$

It follows that $[(p \to q) \vee (q \to p)]_1^f = [p \to q]_1^f \vee [q \to p]_1^f = \frac{1}{2} \vee \frac{1}{2} = \frac{1}{2}$.

Therefore this formula is not valid with respect the new kind of semantics.

6 Forcing value of a formula of MTL

In [Montagna and Ono 2002, Montagna and Sacchetti 2004], it was proved that the r-forcing (this notion was introduced also in [Montagna and Ono 2002] and [Montagna and Sacchetti 2004]) is a more adequate notion for reflecting the logical structure of MTL. Arising from r-forcing, we shall define in this section the $\mathcal{X}$-valued forcing property and forcing value $[\varphi]_{\mathcal{X}}$ of a formula of MTL in a complete MTL-algebra $\mathcal{X}$. The first one is obtained from an $\mathcal{X}$-valued weak forcing property $f : (V \cup \{\perp\}) \times X \to X$ by adding a condition that homogenizes the action of f w.r.t. elements of X. Then one can define the forcing value $[\varphi]_{\mathcal{X}}$, resulting a semantic $[\cdot]_{\mathcal{X}}$ of MTL distinct from $|\cdot|_{\mathcal{X}}$.

One of the main results of the above papers [Montagna and Ono 2002] and [Montagna and Sacchetti 2004] asserts that the Kripke completeness (defined by means of r-forcing) coincides with the usual algebraic completeness of MTL. In this section we shall extend this result, by proving that $[\varphi]_{\mathcal{X}} = \|\varphi\|_{\mathcal{X}}$, for any formula of MTL.

We fix a complete MTL-algebra $\mathcal{X} = (X, \vee, \wedge, \cdot, \to, 0, 1)$.

Definition 12. An **$\mathcal{X}$-valued forcing property** is an $\mathcal{X}$-valued weak forcing property $f : (V \cup \{\perp\}) \times X \to X$ such that $f(\varphi, x) = x \to f(\varphi, 1)$, for any $\varphi \in V$ and $x \in X$.

Definition 13. The **forcing value** $[\varphi]_{\mathcal{X}}$ of a formula φ in $\mathcal{X}$ is defined by

$$[\varphi]_{\mathcal{X}} = \bigwedge\{[\varphi]_1^f \mid \text{f is an } \mathcal{X}\text{-valued forcing property }\}.$$

Let f be an $\mathcal{X}$-valued forcing property.

Proposition 14. *For any* $\varphi \in Form$ *and* $x \in X$, $[\varphi]^f_x = x \rightarrow [\varphi]^f_1$.

Proof. By induction on the complexity of φ.

(1) If φ is an atomic formula, then we apply Definition 12 and we are done.

(2) $[\bot]_x = \overline{x} = x \rightarrow 0 = x \rightarrow [\bot]_1$.

(3) $\varphi = \alpha \vee \beta$. By induction hypothesis, $[\alpha]_x = x \rightarrow [\alpha]_1$ and $[\beta]_x = x \rightarrow [\beta]_1$, hence, by Lemma 3, we get

$$[\varphi]_x = [\alpha]_x \vee [\beta]_x = (x \rightarrow [\alpha]_1) \vee (x \rightarrow [\beta]_1) \leq x \rightarrow ([\alpha]_1 \vee [\beta]_1) = \\ = x \rightarrow [\alpha \vee \beta]_1 = x \rightarrow [\varphi]_1.$$

(4) $\varphi = \alpha \wedge \beta$. By induction hypothesis, $[\alpha]_x = x \rightarrow [\alpha]_1$ and $[\beta]_x = x \rightarrow [\beta]_1$, hence, using Lemma 2, (2), it follows that

$$[\varphi]_x = [\alpha]_x \wedge [\beta]_x = (x \rightarrow [\alpha]_1) \wedge (x \rightarrow [\beta]_1) = x \rightarrow ([\alpha]_1 \wedge [\beta]_1) = \\ = x \rightarrow [\alpha \wedge \beta]_1 = x \rightarrow [\varphi]_1.$$

(5) $\varphi = \alpha \odot \beta$.

By induction hypothesis, $[\alpha]_u = u \rightarrow [\alpha]_1$ and $[\beta]_u = u \rightarrow [\beta]_1$,for all $u \in X$.

Then $[\varphi]_x = \bigvee_{y,z \in x}(x \rightarrow yz)\, [\alpha]_y\, [\beta]_z = \bigvee_{y,z \in x}(x \rightarrow yz)\, (y \rightarrow [\alpha]_1)\, (z \rightarrow [\beta]_1)$.

Let $y, z \in X$. Hence, by Lemma 1, (5),

$x(x \rightarrow yz)\, (y \rightarrow [\alpha]_1)\, (z \rightarrow [\beta]_1) \leq yz\, (y \rightarrow [\alpha]_1)\, (z \rightarrow [\beta]_1) \leq [\alpha]_1\, [\beta]_1$

Therefore, by Lemma 1, (1), we get

$(x \rightarrow yz)\, (y \rightarrow [\alpha]_1)\, (z \rightarrow [\beta]_1) \leq x \rightarrow [\alpha]_1\, [\beta]_1$

This last inequality holds for all $y, z \in X$, therefore

(a) $[\varphi]_x \leq x \rightarrow [\alpha]_1\, [\beta]_1$

Particulary, $[\varphi]_1 \leq [\alpha]_1\, [\beta]_1$. On the other hand,

$[\alpha]_1\, [\beta]_1 = (1 \rightarrow [\alpha]_1\, [\beta]_1)\, ([\alpha]_1 \rightarrow [\alpha]_1)\, ([\beta]_1 \rightarrow [\beta]_1) \leq [\alpha \odot \beta]_1 = [\varphi]_1$

It follows that

(b) $[\varphi]_1 = [\alpha \odot \beta]_1 = [\alpha]_1\, [\beta]_1$

From *(a)* and *(b)* we infer that

(c) $[\varphi]_x \leq x \rightarrow [\varphi]_1$

The converse inequality $x \rightarrow [\varphi]_1 \leq [\varphi]_x$ follows easily by

$x \rightarrow [\varphi]_1 = x \rightarrow [\alpha]_1\, [\beta]_1 = (x \rightarrow [\alpha]_1\, [\beta]_1)\, ([\alpha]_1 \rightarrow [\alpha]_1)\, ([\beta]_1 \rightarrow [\beta]_1) \leq [\varphi]_x$

(6) $\varphi = \alpha \rightarrow \beta$.

By induction hypothesis, $[\alpha]_u = u \rightarrow [\alpha]_1$ and $[\beta]_u = u \rightarrow [\beta]_1$, for all $u \in X$.

Then, by Lemma 1, (6), we get

$$[\varphi]_x = \bigwedge_{y \in X}([\alpha]_y \rightarrow [\beta]_{xy}) = \bigwedge_{y \in X}((y \rightarrow [\alpha]_1) \rightarrow (xy \rightarrow [\beta]_1)) = \\ = \bigwedge_{y \in X}(xy\, (y \rightarrow [\alpha]_1) \rightarrow [\beta]_1)$$

Thus $[\varphi]_x \leq x\ [\alpha]_1 \rightarrow [\beta]_1$. Particulary, $[\varphi]_1 \leq 1\ [\alpha]_1 \rightarrow [\beta]_1$.
For any $y \in X$, we have $y(y \rightarrow [\alpha]_1)\ ([\alpha]_1 \rightarrow [\beta]_1) \leq [\beta]_1$, hence
$[\alpha]_1 \rightarrow [\beta]_1 \leq y(y \rightarrow [\alpha]_1) \rightarrow [\beta]_1 = (y \rightarrow [\alpha]_1) \rightarrow (y \rightarrow [\beta]_1)$
Therefore $[\alpha]_1 \rightarrow [\beta]_1 \leq \bigwedge_{y \in X}((y \rightarrow [\alpha]_1) \rightarrow (y \rightarrow [\beta]_1)) = [\alpha \rightarrow \beta]_1 = [\varphi]_1$.
It follows that
(d) $[\varphi]_1 = [\alpha \rightarrow \beta]_1 = [\alpha]_1 \rightarrow [\beta]_1$,
hence $[\varphi]_x \leq x \rightarrow [\varphi]_1$. On the other hand, by using Lemma 1, (7), we obtain
$x \rightarrow [\varphi]_1 = x \rightarrow ([\alpha]_1 \rightarrow [\beta]_1) = x[\alpha]_1 \rightarrow [\beta]_1 \leq xy\ (y \rightarrow [\alpha]_1) \rightarrow [\beta]_1 =$
$= (y \rightarrow [\alpha]_1) \rightarrow (xy \rightarrow [\beta]_1) = [\alpha]_y \rightarrow [\beta]_{xy}$
Then $x \rightarrow [\varphi]_1 \leq \bigwedge_{y \in X}([\alpha]_y \rightarrow [\beta]_{xy}) = [\varphi]_x$.
We conclude that $[\varphi]_x = x \rightarrow [\varphi]_1$.

Corollary 15. *Let f be an $\mathcal{X}$-valued forcing property. For any $\varphi, \psi \in Form$ we have:*

(1) $[\varphi \vee \psi]_1^f = [\varphi]_1^f \vee [\psi]_1^f$;

(2) $[\varphi \wedge \psi]_1^f = [\varphi]_1^f \wedge [\psi]_1^f$;

(3) $[\varphi \odot \psi]_1^f = [\varphi]_1^f \cdot [\psi]_1^f$;

(4) $[\varphi \rightarrow \psi]_1^f = [\varphi]_1^f \rightarrow [\psi]_1^f$.

Proof. By the proof of Proposition 14.

For any $\mathcal{X}$-valued forcing property f, let us consider the evaluation $\lambda_f : V \rightarrow X$ defined by $\lambda_f(\varphi) = f(\varphi, 1)$, for any $\varphi \in V$.

Proposition 16. *For any $\varphi \in Form$, we have $[\varphi]_1 = \hat{\lambda}_f(\varphi)$.*

Proof. By induction on the complexity of φ, according to Corollary 15

If $e : V \rightarrow X$ is an evaluation, then we define the function
$f_e : (V \cup \{\perp\}) \times X \rightarrow X$ by $f_e(\varphi, x) = x \rightarrow e(\varphi)$, for all $\varphi \in V \cup \{\perp\}$ and $x \in X$. By definition, f_e is a $\mathcal{X}$-valued forcing property.

Proposition 17. *Let $f = f_e$ the $\mathcal{X}$-valued forcing property associated with the evaluation e. For all $\varphi \in Form$ and $x \in X$, we have $[\varphi]_x^f = x \rightarrow \hat{e}(\varphi)$.*

Proof. By induction on the complexity of φ:

- φ is an atomic formula: $[\varphi]_x^f = f(\varphi, x) = x \rightarrow e(\varphi) = x \rightarrow \hat{e}(\varphi)$;
- $\varphi = \alpha \vee \beta$: by induction hypothesis, $[\alpha]_x^f = x \rightarrow \hat{e}(\alpha)$, $[\beta]_x^f = x \rightarrow \hat{e}(\beta)$.

  Then, by using Lemma 2, (4), we obtain

  $[\varphi]_x^f = [\alpha]_x^f \vee [\beta]_x^f = (x \rightarrow \hat{e}(\alpha)) \vee (x \rightarrow \hat{e}(\beta)) \leq$

$\leq x \to (\hat{e}(\alpha) \vee \hat{e}(\beta)) = x \to \hat{e}(\varphi)$

By Lemma 3, (2), we obtain

$$x \to \hat{e}(\varphi) = x \to \hat{e}(\alpha \vee \beta) = x \to (\hat{e}(\alpha) \vee \hat{e}(\beta)) =$$
$$= (x \to \hat{e}(\alpha)) \wedge (x \to \hat{e}(\beta)) = [\alpha]^f_x \wedge [\beta]^f_x \leq [\alpha]^f_x \vee [\beta]^f_x =$$
$$= [\alpha \vee \beta]^f_x = [\varphi]^f_x$$

Therefore $[\varphi]^f_x = x \to \hat{e}(\varphi)$.

- the case $\varphi = \alpha \wedge \beta$ follows similarly;
- $\varphi = \alpha \odot \beta$: by definition and induction hypothesis $[\alpha]^f_x = x \to \hat{e}(\alpha)$, $[\beta]^f_x = x \to \hat{e}(\beta)$, we get

  $[\varphi]^f_x = \bigvee_{y,z \in X}(x \to yz)[\alpha]^f_y\, [\beta]^f_z = \bigvee_{y,z \in X}(x \to yz)(y \to \hat{e}(\alpha))(z \to \hat{e}(\beta))$

  Let $y, z \in X$. Then $x(x \to yz)(y \to \hat{e}(\alpha))(z \to \hat{e}(\beta)) \leq \hat{e}(\alpha)\hat{e}(\beta)$, hence $(x \to yz)(y \to \hat{e}(\alpha))(z \to \hat{e}(\beta)) \leq x \to \hat{e}(\alpha \odot \beta) = x \to \hat{e}(\varphi)$. It results that $[\varphi]^f_x \leq x \to \hat{e}(\varphi)$. According to the previous expression of $[\varphi]^f_x$, the converse inequality $x \to \hat{e}(\varphi) = x \to \hat{e}(\alpha)\hat{e}(\beta) \leq [\varphi]^f_x$ is obvious;
- $\varphi = \alpha \to \beta$: by induction hypothesis, $[\alpha]^f_u = u \to \hat{e}(\alpha)$, $[\beta]^f_u = u \to \hat{e}(\beta)$, for all $u \in X$. According to Lemma 2, (2), we can write

  $$[\varphi]^f_x = \bigwedge_{y \in X}([\alpha]_y \to [\beta]_{xy}) = \bigwedge_{y \in X}((y \to \hat{e}(\alpha)) \to (xy \to \hat{e}(\beta))) =$$
  $$= \bigwedge_{y \in X}(x \to ((y \to \hat{e}(\alpha)) \to (y \to \hat{e}(\beta)))) =$$
  $$= x \to \bigwedge_{y \in X}((y \to \hat{e}(\alpha)) \to (y \to \hat{e}(\beta)))$$

  Thus $[\varphi]^f_x \leq x \to ((1 \to \hat{e}(\alpha)) \to (1 \to \hat{e}(\beta))) = x \to (\hat{e}(\alpha) \to \hat{e}(\beta)) = x \to \hat{e}(\varphi)$. Let $y \in X$. Then $y(y \to \hat{e}(\alpha))(\hat{e}(\alpha) \to \hat{e}(\beta)) \leq \hat{e}(\beta)$, hence

  $$\hat{e}(\alpha \to \beta) = \hat{e}(\alpha) \to \hat{e}(\beta) \leq y(y \to \hat{e}(\alpha)) \to \hat{e}(\beta) =$$
  $$= (y \to \hat{e}(\alpha)) \to (y \to \hat{e}(\beta))$$

  This inequality is true for any $y \in X$, so

  $$\hat{e}(\alpha \to \beta) \leq \bigwedge_{y \in X}((y \to \hat{e}(\alpha)) \to (y \to \hat{e}(\beta)))$$

  Applying Lemma 1, (7), we obtain

  $$x \to \hat{e}(\varphi) = x \to \hat{e}(\alpha \to \beta) \leq x \to \bigwedge_{y \in X}((y \to \hat{e}(\alpha)) \to (y \to \hat{e}(\beta))) = [\varphi]^f_x$$

Proposition 18. *There exists a bijective correspondence between the $\mathcal{X}$-valued forcing properties and the evaluations of MTL in $\mathcal{X}$.*

Proof. The assignments $f \mapsto \lambda_f$ and $e \mapsto f_e$ prove the bijective correspondence between the set of $\mathcal{X}$-valued forcing properties and the set of evaluations in $\mathcal{X}$.

The following theorem is a consequence of the previous results.

Theorem 19. *For any formula φ of MTL, we have $[\varphi]_{\mathcal{X}} = \|\varphi\|_{\mathcal{X}}$.*

Corollary 20. *If the formula φ is provable in MTL, then $[\varphi]_{\mathcal{X}} = 1$.*

7 Final discussion and open questions

We shall discuss two possible directions to extend and improve the results obtained in the previous sections.

7.1 The predicate logic $MTL\forall$ was introduced by Esteva and Godo in the paper [Esteva and Godo 2001]. The language of $MTL\forall$ has the following primitive symbols: variables, predicates symbols, the connectives $\vee, \wedge, \odot, \rightarrow$, the constant $\bot$, the quantifiers $\exists, \forall$ and the paranthesis $(,)$. The axioms of $MTL\forall$ are those of MTL plus:

(∀1) $\forall v\, \varphi \rightarrow \varphi(w/v)$ (w is substitutable for v in φ)

(∀2) $\forall v\, (\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \forall v\, \psi)$ (v is not free in φ)

(∀3) $\forall v\, (\varphi \vee \psi) \rightarrow (\varphi \vee \forall v\, \psi)$ (v is not free in φ)

(∃1) $\varphi(w/v) \rightarrow \exists v\, \varphi$ (w is substitutable for v in φ)

(∃2) $\forall v\, (\varphi \rightarrow \psi) \rightarrow (\exists v\, \varphi \rightarrow \psi)$ (v is not free in ψ).

The inference rules of $MTL\forall$ are modus ponens and generalization: $\frac{\varphi}{\forall x\, \varphi}$.

The formulas and the sentences of $MTL\forall$ are defined as usual. If D is a non-empty set, then $MTL\forall(D)$ will be the language obtained from $MTL\forall$ by adding the elements of D as new constants.

Let $\mathcal{X}$ be a complete MTL-algebra and D a non-empty set. A first-order $\mathcal{X}$-evaluation with domain D is a function e from the set $At(D)$ of atomic sentences in $MTL\forall(D)$ into $\mathcal{X}$. Any first-order $\mathcal{X}$-evaluation e with domain D can be uniquely extended by induction to a function $\hat{e}$ from the sentences of $MTL\forall(D)$ into $\mathcal{X}$. The truth value $\|\varphi\|_{\mathcal{X}}$ of a sentence φ of $MTL\forall(D)$ in $\mathcal{X}$ is defined as usual [Esteva and Godo 2001, Esteva et.al. 2002].

Now we shall extend the definitions of preceding sections to the new setting. An $\mathcal{X}$-valued weak forcing property with domain D is a function $f : (At(D) \cup \{\bot\}) \times X \rightarrow X$ such that $f(\bot, 1) = 0$ and, for all $\varphi \in At(D)$ and $x, y \in X$, $x \leq y$ implies $f(\varphi, y) \leq f(\varphi, x)$. In an analogous way we can define the notion of $\mathcal{X}$-valued forcing property with domain D.

Let f be an $\mathcal{X}$-valued weak forcing property with domain D. For any sentence φ of $MTL\forall(D)$ and $x \in X$, the element $[\varphi]^f_x$ of X is defined by the conditions (1)-(6) of Definition 5 and the following new clauses:

(i) If $\varphi = \forall v\, \psi$, then $[\varphi]^f_x = \bigwedge_{d \in D} [\psi(d)]^f_x$;

(ii) If $\varphi = \exists v\, \psi$, then $[\varphi]^f_x = \bigwedge_{y<x} \bigvee_{y<z} \bigvee_{d \in D} [\psi(d)]^f_z$.

Now, for any sentence φ of $MTL\forall$, we can define the weak forcing value $|\varphi|_{\mathcal{X}}$ and the forcing value $[\varphi]_{\mathcal{X}}$ of φ in $\mathcal{X}$.

For $|\cdot|_{\mathcal{X}}$ and $[\cdot]_{\mathcal{X}}$ we can formulate the following open questions:

Open question 21. Analyse the behaviour of $|\cdot|_{\mathcal{X}}$ and $[\cdot]_{\mathcal{X}}$ w.r.t. the axioms and some other types of sentences in $MTL\forall$.

Open question 22. Compare the semantics $|\cdot|_{\mathcal{X}}$, $[\cdot]_{\mathcal{X}}$, $\|\cdot\|_{\mathcal{X}}$ and extend the results of Section 5.

The following two propositions constitute a first step in solving the problem 21. We fix a $\mathcal{X}$-valued weak forcing property f with domain D.

Proposition 23. *Let $\varphi(v)$ be a formula of $MTL\forall$, χ a sentence of $MTL\forall$, $x \in X$ and $a \in D$. Then the following properties hold:*

(1) $[\forall v\, \varphi]^f_x \leq [\varphi(a)]^f_x$;

(2) $[\varphi(a)]^f_x \leq [\exists v\, \varphi]^f_x$;

(3) $[\forall v\, (\chi \to \varphi)]^f_x = [\chi \to \forall v\, \varphi]^f_x$;

(4) $[\exists v\, \varphi \to \chi]^f_x = [\forall v\, (\varphi \to \chi)]^f_x$.

Proof.
(1) Obvious.

(2) For any $y < x$, we have$[\varphi(a)]^f_x \leq \bigvee_{y<z} \bigvee_{b\in D} [\varphi(b)]^f_z$, hence $[\varphi(a)]^f_x \leq \bigwedge_{y<x} \bigvee_{y<z} \bigvee_{b\in D} [\varphi(b)]^f_z = [\exists v\, \varphi]^f_x$.

(3) By the definition of $[\cdot]^f_x$ and Lemma 2, (2), we get $[\forall v\, (\chi \to \varphi)]^f_x = \bigwedge_{b\in D} \bigwedge_{y\in X} ([\chi]^f_y \to [\varphi(b)]^f_{xy}) = \bigwedge_{y\in X}([\chi]^f_y \to \bigwedge_{b\in D}[\varphi(b)]^f_{xy}) = \bigwedge_{y\in X}([\chi]^f_y \to [\forall v \varphi]^f_{xy})$ $= [\chi \to \forall v\, \varphi]^f_x$.

(4) Let $b \in D$ and $y \in X$. According to Proposition 8, (4) and the previous inequality *(2)* we get $[\exists v \varphi \to \chi]^f_x \leq [\exists v \varphi]^f_y \to [\chi]^f_{xy} \leq [\varphi(b)]^f_y \to [\chi]^f_{xy}$. Therefore, for any $b \in D$, we have $[\exists v\, \varphi \to \psi]^f_x \leq \bigwedge_{y\in X}(\, [\varphi(b)]^f_y \to [\chi]^f_{xy}\,) = [\varphi(b) \to \chi]^f_x$. Thus $[\exists v\, \varphi \to \chi]^f_x \leq \bigwedge_{b\in D} [\varphi(b) \to \chi]^f_x = [\forall v\, (\varphi \to \chi)]^f_x$.

Proposition 24. *Let $\varphi(v)$, $\psi(v)$ two formulas of $MTL\forall$. Then $[\forall v\, (\varphi \to \psi)]^f_x \leq [\forall v\, \varphi \to \forall v\, \psi]^f_x$.*

Proof. Let $y \in X$ and $a \in D$. By Proposition 23, (1), and Proposition 8, (4), we get $[\forall v\, (\varphi \to \psi)]^f_x \cdot [\forall v\, \varphi]^f_y \leq [\varphi(a) \to \psi(a)]^f_x \cdot [\varphi(a)]^f_y \leq [\psi(a)]^f_{xy}$, hence $[\forall v\, (\varphi \to \psi)]^f_x \cdot [\forall v\, \varphi]^f_y \leq \bigwedge_{a\in D} [\psi(a)]^f_{xy}$. Thus $[\forall v\, (\varphi \to \psi)]^f_x \leq [\forall v\, \varphi]^f_y \to [\forall v\, \psi]^f_{xy}$, for each $y \in X$. Therefore, $[\forall v\, (\varphi \to \psi)]^f_x \leq \bigwedge_{y\in X}([\forall v\, \varphi]^f_y \to [\forall v\, \psi]^f_{xy})$ $= [\forall v\, \varphi \to \forall v\, \psi]^f_x$.

7.2 Recently, a lot of non-commutative fuzzy algebras and their logical calculi were investigated [Cintula and Hájek 2006], [Gottwald 2005], [Iorgulescu 2006a], [Iorgulescu 2006b], [Piciu 2007]. Pseudo MTL-algebras (psMTL-algebras, for short) were defined in [Flondor et.al. 2001] arising from the structure of the interval $[0, 1]$ induced by a left-continuous non-commutative t-norm.

A *ps*MTL*-algebra* is a structure $\mathcal{X} = (X, \vee, \wedge, \cdot, \rightarrow, \rightsquigarrow, 0, 1)$, where:

(C1) $(X, \vee, \wedge, 0, 1)$ is a bounded lattice;

(C2) $(X, \cdot, 1)$ is a monoid;

(C3) $x \cdot y \leq z$ iff $x \leq y \rightarrow z$ iff $y \leq x \rightsquigarrow z$;

(C4) $(x \rightarrow y) \vee (y \rightarrow x) = (x \rightsquigarrow y) \vee (y \rightsquigarrow x) = 1$.

By definition, a psMTL-algebra $\mathcal{X}$ is *representable* if it is isomorphic to a subdirect product of psMTL-chains [3] The variety of representable psMTL-algebras is characterized by Kühr's identities [Kühr 2003]:

$$(y \rightarrow z) \vee (z \rightsquigarrow ((x \rightarrow y) \cdot z)) = 1$$
$$(y \rightsquigarrow z) \vee (z \rightarrow (z \cdot (x \rightsquigarrow y))) = 1$$

The psMTL-algebras constitute the algebraic base for the propositional calculul psMTL, elaborated in [Hájek 2003a, Hájek 2003b]. An extension of psMTL is psMTL^r, a logical system obtained from psMTL by adding Kühr's axioms:

(K1) $(\psi \rightarrow \varphi) \vee (\chi \rightsquigarrow ((\varphi \rightarrow \psi) \odot \chi))$;

(K2) $(\psi \rightsquigarrow \varphi) \vee (\chi \rightarrow (\chi \odot (\varphi \rightsquigarrow \psi)))$.

A standard completeness theorem for psMTL^r was proved by Jenei and Montagna in [Jenei and Montagna 2003], by using a generalization of a technique from [Jenei and F. Montagna 2002].

Two predicate logics ps$MTL\forall$ and ps$MTL\forall^r$ were developed by Hájek and Ševčik in [Hájek and Ševčik 2004] and an weak completeness theorem for the psMTL^r logic was established.

In the framework of logics psMTL, psMTL^r, ps$MTL\forall$ and ps$MTL\forall^r$ we can formulate the following open questions:

Open question 25. Extend the Kripke semantics of [Montagna and Ono 2002, Montagna and Sacchetti 2004] to these non-commutative logics in order to obtain similar standard completeness theorems for psMTL^r and ps$MTL\forall^r$.

[3] A non-commutative residuated lattice is a structure $\mathcal{X} = (X, \vee, \wedge, \cdot, \rightarrow, \rightsquigarrow, 0, 1)$ verifying the conditions (C1)-(C3) (see [Jipsen and Tsinakis 2002]). Any totally ordered non-commutative residuated lattice is a psMTL-chain.

Open question 26. Define appropiate notions of weak forcing value and forcing value for the logics psMTL, psMTL^r, ps$MTL\forall$, ps$MTL\forall^r$ and obtain non-commutative versions of the results proved in Sections 4 and 5.

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