MOEM

Modern Electronic Materials 2024; 10(2): 103–113
DOI 10.3897/j.moem.10.2.127996

Research Article

To a dispersion law of an electromagnetic wave in a medium having a double anisotropy

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Received 21 May 2024 ♦ **Accepted** 4 June 2024 ♦ **Published** 5 July 2024

Citation: Sobol VR, Silibin MV, Karpinsky DV (2024) To a dispersion law of an electromagnetic wave in a medium having a double anisotropy. *Modern Electronic Materials* 10(2): 103–113. https://doi.org/10.3897/j.moem.10.2.127996

Abstract

Dispersion relations for a transverse electromagnetic wave in a hypothetical medium having a low crystalline symmetry and double anisotropy are formulated phenomenological based on the general concepts of Maxwell's equations. Formalism for electrical and magnetic ordering of a medium is represented by tensor coefficients of the second rank in a general coordinate system corresponding to an external problem of incidence, reflection and refraction of a transverse wave at the interface. Based on the obtained dispersion law, the Snell law of refraction for the wave vector is displayed, as well as, some special cases for a section of a wave surface and a ray surface in the plane of incidence. Following Fresnel approach as usual, two characteristic types of linear polarization of an electric induction vector of a propagating wave are considered for a wave entering a double anisotropic medium at an arbitrary angle to the interface. Based on a character of a cross-section of a wave surface and of a ray surface for the plane of incidence, the degree of influence of anisotropy on a propagating field is discussed in terms of external/internal conic refraction of ordinary/extraordinary waves in a transparent medium. Following to initially general form of the permittivity $\hat{\varepsilon}$ and the magnetic permeability $\hat{\mu}$ tensors some particular cases of a more favorable choice of coordinate system are analyzed to decrease a number of non-zero components in mentioned tensors, as well as to decrease in the number of elements in vector relations.

Keywords

dispersion law, Fresnel problem, refraction, polarized wave

1. Introduction

At present the validity of standard retrieval methods that assign bulk electromagnetic properties, such as a electric permittivity ε and a magnetic permeability μ, from calculations of the scattering (*S*) parameters for finite-thickness samples is widely discussed. Such *S*-parameter retrieval methods have recently become a principal means of characterizing artificially structured materials, which, by nature, are inherently inhomogeneous. While a unit cell of a material can be made considerably smaller than the free space wavelength, there remains a significant variation of the phase across a unit cell at operational frequencies in nearly all metamaterial structures reported to date. In the approach of mentioned above it is indicates that it is conceptually convenient to replace a collection of scattering objects by a homogeneous medium, whose electromagnetic properties result from an averaging of the local responding electromagnetic fields and current distributions. Ideally, there would be no distinction in the observed electromagnetic response of a hypothetical continuous material versus that of the composite it replaces. This equivalence can be readily achieved when the applied fields are static or have spatial variation on a scale significantly larger than the scale of a local inhomogeneity, in which case the composite is said to form an effective medium.

The electromagnetic properties of an inhomogeneous composite can be determined exactly by solving Maxwell's equations, which relate a local electric and magnetic fields to the local charge and current densities. When the particular details of the inhomogeneous structure are unimportant to the behavior of relevant fields of interest, the local field, charge, and current distributions are averaged, yielding a macroscopic form of Maxwell's equations [1, 2]. To solve this set of equations, a relationship must be assumed that relates the four macroscopic field vectors that arise from an averaging – or homogenization – procedure. It is here that an electric permittivity and a magnetic permeability tensors are typically defined, which encapsulate the specific local details of a composite medium [1, 3–7].

Here we discuss a case of homogeneous material on a base of some hypothetical anisotropic medium of crystalline type which has macroscopically ordered properties both of electric and magnetic nature. Electrical ordering in optics in some crystal media is widely known to lead to birefringence (for example crystalline quartz, calcite, aragonite, etc.). It is reasonably to propose that a magnetic ordering also can stimulate some similar behavior of a linearly polarized electromagnetic wave via refraction index that is via a magnetic permeability.

In media having a double type of anisotropy, for example in so called transparent ferroelectrics, it is rather difficult to pose a question of an experimental solution of inverse Fresnel problem that is to restore the properties of a medium based on a reflection experiment. Such medium may have two systems of optical axes of electric and magnetic nature, which as usually said in general do not necessarily coincide in space. Accordingly, it is rather difficult to describe the features of a reflection of polarized light in the form of Fresnel relations. Moreover, it should be noted that even an attempt to describe a reflection of polarized light from an interface with medium having usual type of anisotropy (for example mentioned calcite, aragonite or crystalline quartz etc.) is in a general case far from a trivial task namely for a wave being polarized corresponding to extraordinary type (when a refractive index is a function of propagate angle). In this Fresnel problem, the optical axes are needed to be correlated with an external coordinate system that determines the plane of incidence, angles of incidence, and refraction. But, in fact, for an external task, which is the search for Fresnel coefficients, a main aim is dispersion law, the knowledge of the orientation of a main coordinate axes is not so relevant against a background of correct matching of the field components for incident, reflected and refracted waves. That is, on the contrary, about a medium itself, its type and a nature of its axes, it is quite possible to draw some conclusions based on a recording and analysis of a dispersion law and corresponding Fresnel relations, with additional use of results from measuring of reflection coefficients at accepted refractive conditions on interface.

Within of the mentioned above, a purpose of this work is to express a dispersion law for transverse electro-magnetic wave at two typical linear polarization for electric induction vector with respect to a selected plane, being a plane of incidence. Here we are talking about two Fresnel waves propagation in homogeneous crystalline medium being ordered as electrically as magnetically. One of mentioned wave is linearly polarized in a plane of incidence, and another wave is linearly polarized normally to the plane of incidence.

In this representation, the dispersion law, after writing it down, allows us to formulate the law of refraction for wave vector. Respectively there appears a possibility to obtain an expressions required subsequently to formulate Fresnel incidence/reflection relations at an interface. So a nature of linearly polarized wave incidence/reflection/refraction/propagation it makes sense to formulate and describe via initial dispersion relations in a general coordinate system of dielectric and magnetic tensors. At first we do not use a coordinate system of principal axes but use a general coordinate system of tensor coefficients, when one of a tensor coordinate planes coincides with a plane of incidence of a wave and another plane is that of an interface. As a result, based on a written dispersion law, it will be possible to identify the most rational approaches to experimentally solving an inverse Fresnel problem. That is within the framework of achieved relations and based on the results of measuring a reflection coefficient to restore a type and parameters of tensor material constants.

2. Prerequisites for setting and solving the problem, used approaches

The study of Fresnel reflection, as mentioned, is promising for a double anisotropy research in view of a possibility of identifying magnetic and optical properties based on the results of inverse consideration of a dependence of the measured reflection coefficient and a state of polarization as a function of an angle of incidence. As previously stated, the results of a development of synthesis technologies, materials scientists are successfully mastering a class of new substances being complex in composition and crystal structure. These materials are usually ordered by both electrical and magnetic degrees of freedom and include transparent thin layers of ferromagnets, magnetic semiconductors-ferrites, materials of a so-called ferroelectric segment [8–16]. For example, in materials based on bismuth ferrite and its solid solutions, the properties are formed by a combination of electric and magnetic ordering and their mutual interaction of longitudinal and transverse types. Moreover, each of these mentioned ordering factors is capable of influencing an electromagnetic field because a refractive index is a function of both electric and magnetic permeability. From a point of view of a validity of introducing two tensor coefficients, there are no fundamental prohibitions for describing optical properties including a refracting index by the electrical and magnetic mechanisms. Moreover, the use of two tensor coefficients allows to formulate boundary conditions in a most familiar and accessible form following directly from the principles of Maxwell's electrodynamics [17–22].

In given case, an optical medium having a double type of anisotropy is considered using an example of a model ferroelectric structure having a low level of crystal symmetry, for example, of a rhombohedral type and lower. Here we consider a phenomenological aspect and leave the question of microscopic mechanisms of a formation of double ordering. Does ordering appear due to direct positive/negative exchange, or due to indirect exchange/over exchange of magnetic ions in the crystal structure, or due to other mechanisms. We simply postulate that an optically transparent medium can be characterized by orthogonal polar Hermitian tensors of electric permittivity $\hat{\varepsilon}$ and magnetic permeability $\hat{\mu}$. For generality of consideration, at the very beginning of the analysis, the coordinate system is chosen such that all components of both tensors are nonzero. Based on these considerations there will be shown below that an entire spectrum of known optical anisotropy can be represented through tensor coefficients of a more simpler form with a rational choice of an incidence plane with respect to directions of crystal symmetry of a medium.

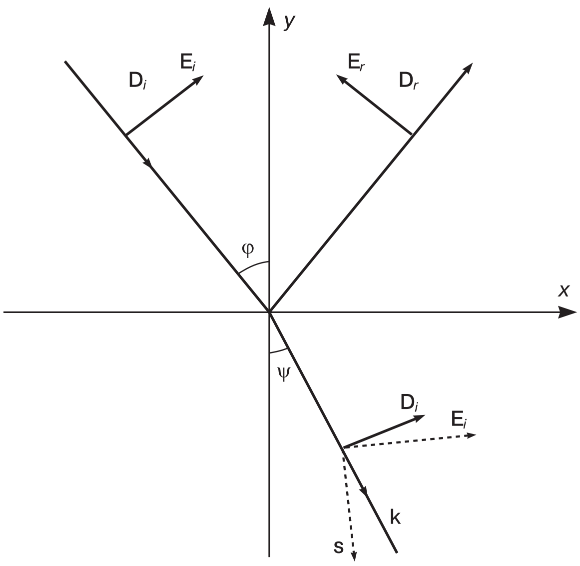

Figure 1. The geometry of an entry of a linearly polarized wave (angle of incidence – φ, angle of refraction – ψ) into the surface (*xz* plane) of an anisotropic optical medium (down), **k** is a wave vector in an anisotropic medium, **s** is a ray vector for one of the special cases, an upper area is an isotropic medium

3. Methodology for the problem. Analysis and discussion of the results

As mentioned above, the crystalline medium in a general coordinate system of accepted dielectric and magnetic tensors $\hat{\varepsilon}$ and $\hat{\mu}$ can be represented in the formalism Maxwell's electrodynamics. Here the material coupling relations include tensor coefficients of the form

$$\hat{\varepsilon} = \begin{pmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{xz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{pmatrix}, \quad \hat{\mu} = \begin{pmatrix} \mu_{xx} & \mu_{xy} & \mu_{xz} \\ \mu_{yx} & \mu_{yy} & \mu_{yz} \\ \mu_{zx} & \mu_{zy} & \mu_{zz} \end{pmatrix}. \quad (1)$$

The known relationships between the components of the vectors of electrical and magnetic induction (**D** and **B**) and a strength (**E** and **H**) for a transverse wave (in a SI system of units), as usual, have a form (2), (3):

$$\mathbf{H} = \mu_0^{-1}\hat{\mu}^{-1}\mathbf{B}, \quad (2)$$

$$\mathbf{E} = \varepsilon_0^{-1}\hat{\varepsilon}^{-1}\mathbf{D}, \quad (3)$$

here ε_0 and μ_0 are an electric and magnetic constants, symbols like $\hat{\varepsilon}^{-1}$ etc. – denote inverse tensors/their components.

The geometry of a problem, presented in a tensor coordinate system, is represented for a linearly polarized wave as it enters a medium according to an incidence-reflection-refraction scenario (Fig. 1).

In Fig. 1 the *y* axis is an outer normal to an interface, a wave along an electric induction vector is polarized in a plane of incidence *xy*, *xy*-polarization. The upper medium is isotropic one. The problem also considers a case of linear polarization oriented normal to plane *xy*, *z*-polarization. Wave vectors are accepted as real. All these, for an incident (wave vector is not marked), for a reflected (wave vector is not marked) and for refracted (wave vector is denoted as **k**) lie in a *xy* plane (4).

$$k_x = k\sin\psi,\ k_y = -k\cos\psi,\ k_z = 0. \quad (4)$$

The condition of continuity of the normal components of an electrical induction vector at an interface leads to fact that when a transverse electromagnetic wave falls from an isotropic medium onto the interface with an anisotropic medium, the tangential component of a wave vector remains continuous and a wave vector in a *xy* plane will also have after refraction a zero *z*-component of a form **k** ($k_x : k_y : k_z = 0$).

To formulate the relation for a dispersion expression of a refracted wave, we use Maxwell's equations and respective tensor equation (5)

$$-\nabla\times\hat{\mu}^{-1}\nabla\times\hat{\varepsilon}^{-1}\mathbf{D}=\varepsilon_0\mu_0\ddot{\mathbf{D}}, \tag{5}$$

In a basis of unit vectors **i**, **j**, **k** of chosen coordinate system being, as mentioned, the same for tensors at wave reflection/refraction, the elements of equation (6) for the components of a vector **D** in a transverse linearly polarized wave in accepted most general case of arbitrary orientation of the vectors **D** and **k** satisfy the standard relations (6)–(9)

$$\mathbf{D}=(D_x\mathbf{i}+D_y\mathbf{j}+D_z\mathbf{k})e^{i(\omega t-\mathbf{kr})}, \tag{6}$$

$$\mathbf{kr}=k_x x+k_y y+k_z z, \tag{7}$$

$$\frac{\partial D_x}{\partial x}=-ik_x D_x e^{\mathrm{i}(\omega t-\mathbf{kr})}, \tag{8}$$

$$\frac{\partial D_x}{\partial y}=-ik_y D_x e^{\mathrm{i}(\omega t-\mathbf{kr})},\ \text{etc.} \tag{9}$$

Here we emphasize for clarity that **i**, **j**, **k** denote a unit vectors-orts of coordinate system, **k** (k_x, k_y, k_z) is a wave vector and its components, ***i*** is an imaginary unit. Following (6)–(9), tensor relation (5) can be reduced to a (10)

$$\mathbf{k}\times\hat{\mu}^{-1}\mathbf{k}\times\hat{\varepsilon}^{-1}\mathbf{D}=\varepsilon_0\mu_0\ddot{\mathbf{D}}, \tag{10}$$

After some detalization (10) is transformed to (11)

$$\mathbf{k}\times\hat{\mu}^{-1}\mathbf{k}\times\varepsilon_{ml}^{-1}\mathbf{D}_l=\varepsilon_0\mu_0\ddot{\mathbf{D}}, \tag{11}$$

here, by coinciding indices, is accepted, as usual, summation.

Next steps leads to (12)

$$\begin{aligned}&\mathbf{k}\left\{\hat{\mu}^{-1}\mathbf{k}\left[\mathbf{i}\left(\varepsilon_{xl}^{-1}D_l\right)+\mathbf{j}\left(\varepsilon_{yl}^{-1}D_l\right)+\mathbf{k}\left(\varepsilon_{zl}^{-1}D_l\right)\right]\right\}=\\&=\varepsilon_0\mu_0\ddot{\mathbf{D}}.\end{aligned} \tag{12}$$

Such for a component of D_x for electrical induction vector a relation in general view is (13)

$$\begin{aligned}&k_z\Big[\mu_{yx}^{-1}\big(k_y\varepsilon_{zm}^{-1}D_m-k_z\varepsilon_{ym}^{-1}D_m\big)+\mu_{yy}^{-1}\big(k_z\varepsilon_{xm}^{-1}D_m-\\&-k_x\varepsilon_{zm}^{-1}D_m\big)+\mu_{yz}^{-1}\big(k_x\varepsilon_{ym}^{-1}D_m-k_y\varepsilon_{xm}^{-1}D_m\big)\Big]-\\&-k_y\Big[\mu_{zx}^{-1}\big(k_y\varepsilon_{zm}^{-1}D_m-k_z\varepsilon_{ym}^{-1}D_m\big)+\mu_{zy}^{-1}\big(k_z\varepsilon_{xm}^{-1}D_m-\\&-k_x\varepsilon_{zm}^{-1}D_m\big)+\mu_{zz}^{-1}\big(k_x\varepsilon_{ym}^{-1}D_m-k_y\varepsilon_{xm}^{-1}D_m\big)\Big]=\\&=\varepsilon_0\mu_0\omega^2D_x.\end{aligned} \tag{13}$$

In (13) there presented a most general case of transverse wave propagation through crystalline medium of a type (1). Here a wave vector **k** is not only **k** (k_x : k_y : $k_z=0$), but may have any components **k** (k_x : k_y : k_z). In fact, the equations (12), (13) in coordinate representation give a system of equations that contains three lines (14)–(16), namely:

First line (14) is

$$\begin{aligned}&\Big[-\mu_{zx}^{-1}\big(\varepsilon_{zx}^{-1}k_y^2-\varepsilon_{yx}^{-1}k_zk_y\big)-\mu_{zy}^{-1}\big(\varepsilon_{xx}^{-1}k_zk_y-\\&-\varepsilon_{zx}^{-1}k_xk_y\big)-\mu_{zz}^{-1}\big(\varepsilon_{yx}^{-1}k_xk_y-\varepsilon_{xx}^{-1}k_y^2\big)+\\&+\mu_{yx}^{-1}\big(\varepsilon_{zx}^{-1}k_zk_y-\varepsilon_{yx}^{-1}k_z^2\big)+\mu_{yy}^{-1}\big(\varepsilon_{xx}^{-1}k_z^2-\varepsilon_{zx}^{-1}k_xk_z\big)+\\&+\mu_{yz}^{-1}\big(\varepsilon_{yx}^{-1}k_xk_z-\varepsilon_{xx}^{-1}k_zk_y\big)-\varepsilon_0\mu_0\omega^2\Big]D_x+\\&+\Big[-\mu_{zx}^{-1}\big(\varepsilon_{zy}^{-1}k_y^2-\varepsilon_{yy}^{-1}k_zk_y\big)-\mu_{zy}^{-1}\big(\varepsilon_{xy}^{-1}k_zk_y-\\&-\varepsilon_{zy}^{-1}k_xk_y\big)-\mu_{zz}^{-1}\big(\varepsilon_{yy}^{-1}k_xk_y-\varepsilon_{xy}^{-1}k_y^2\big)+\\&+\mu_{yx}^{-1}\big(\varepsilon_{zy}^{-1}k_zk_y-\varepsilon_{yy}^{-1}k_z{}^2\big)+\mu_{yy}^{-1}\big(\varepsilon_{xy}^{-1}k_z^2-\\&-\varepsilon_{zy}^{-1}k_xk_z\big)+\mu_{yz}^{-1}\big(\varepsilon_{yy}^{-1}k_xk_z-\varepsilon_{xy}^{-1}k_zk_y\big)\Big]D_y+\\&+\Big[-\mu_{zx}^{-1}\big(\varepsilon_{zz}^{-1}k_y^2-\varepsilon_{yz}^{-1}k_zk_y\big)-\mu_{zy}^{-1}\big(\varepsilon_{xz}^{-1}k_zk_y-\\&-\varepsilon_{zz}^{-1}k_xk_y\big)-\mu_{zz}^{-1}\big(\varepsilon_{yz}^{-1}k_xk_y-\varepsilon_{xz}^{-1}k_y^2\big)+\\&+\mu_{yx}^{-1}\big(\varepsilon_{zz}^{-1}k_zk_y-\varepsilon_{yz}^{-1}k_z^2\big)+\mu_{yy}^{-1}\big(\varepsilon_{xz}^{-1}k_z^2-\\&-\varepsilon_{zz}^{-1}k_xk_z\big)+\mu_{yz}^{-1}\big(\varepsilon_{yz}^{-1}k_xk_z-\varepsilon_{xz}^{-1}k_zk_y\big)\Big]D_z=0.\end{aligned} \tag{14}$$

Second line (15) is

$$\begin{aligned}&\Big[-\mu_{xx}^{-1}\big(\varepsilon_{zx}^{-1}k_zk_y-\varepsilon_{yx}^{-1}k_z^2\big)-\mu_{xy}^{-1}\big(\varepsilon_{xx}^{-1}k_z^2-\\&-\varepsilon_{zx}^{-1}k_xk_z\big)-\mu_{xz}^{-1}\big(\varepsilon_{yx}^{-1}k_xk_z-\varepsilon_{xx}^{-1}k_zk_y\big)+\\&+\mu_{zx}^{-1}\big(\varepsilon_{zx}^{-1}k_xk_y-\varepsilon_{yx}^{-1}k_xk_z\big)+\mu_{zy}^{-1}\big(\varepsilon_{xx}^{-1}k_xk_z-\\&-\varepsilon_{zx}^{-1}k_x^2\big)+\mu_{zz}^{-1}\big(\varepsilon_{yx}^{-1}k_x^2-\varepsilon_{xx}^{-1}k_xk_y\big)\Big]D_x+\\&+\Big[-\mu_{xx}^{-1}\big(\varepsilon_{zy}^{-1}k_zk_y-\varepsilon_{yy}^{-1}k_z^2\big)-\mu_{xy}^{-1}\big(\varepsilon_{xy}^{-1}k_z^2-\\&-\varepsilon_{zy}^{-1}k_xk_z\big)-\mu_{xz}^{-1}\big(\varepsilon_{yy}^{-1}k_xk_z-\varepsilon_{xy}^{-1}k_zk_y\big)+\\&+\mu_{zx}^{-1}\big(\varepsilon_{zy}^{-1}k_xk_y-\varepsilon_{yy}^{-1}k_xk_z\big)+\mu_{zy}^{-1}\big(\varepsilon_{xy}^{-1}k_xk_z-\\&-\varepsilon_{zy}^{-1}k_x^2\big)+\mu_{zz}^{-1}\big(\varepsilon_{yy}^{-1}k_x^2-\varepsilon_{xy}^{-1}k_xk_y\big)-\varepsilon_0\mu_0\omega^2\Big]D_y+\\&+\Big[-\mu_{xx}^{-1}\big(\varepsilon_{zz}^{-1}k_zk_y-\varepsilon_{yz}^{-1}k_z^2\big)-\mu_{xy}^{-1}\big(\varepsilon_{xz}^{-1}k_z^2-\\&-\varepsilon_{zz}^{-1}k_xk_z\big)-\mu_{xz}^{-1}\big(\varepsilon_{yz}^{-1}k_xk_z-\varepsilon_{xz}^{-1}k_zk_y\big)+\\&+\mu_{zx}^{-1}\big(\varepsilon_{zz}^{-1}k_xk_y-\varepsilon_{yz}^{-1}k_zk_x\big)+\mu_{zy}^{-1}\big(\varepsilon_{xz}^{-1}k_xk_y-\\&-\varepsilon_{zz}^{-1}k_x^2\big)+\mu_{zz}^{-1}\big(\varepsilon_{yz}^{-1}k_z^2-\varepsilon_{xz}^{-1}k_xk_y\big)\Big]D_z=0.\end{aligned} \tag{15}$$

Third line (16) is

$$
\begin{aligned}
&\Big[-\mu_{yx}^{-1}\left(\varepsilon_{zx}^{-1}k_xk_y-\varepsilon_{yx}^{-1}k_xk_z\right)-\mu_{yy}^{-1}\left(\varepsilon_{xx}^{-1}k_xk_z-\right.\\
&\left.-\varepsilon_{zx}^{-1}k_x^2\right)-\mu_{yz}^{-1}\left(\varepsilon_{yx}^{-1}k_x^2-\varepsilon_{xx}^{-1}k_xk_y\right)+\\
&\mu_{xx}^{-1}\left(\varepsilon_{zx}^{-1}k_y^2-\varepsilon_{yx}^{-1}k_xk_z\right)+\mu_{xy}^{-1}\left(\varepsilon_{xx}^{-1}k_yk_z-\right.\\
&\left.-\varepsilon_{zx}^{-1}k_xk_y\right)+\mu_{xz}^{-1}\left(\varepsilon_{yx}^{-1}k_xk_y-\varepsilon_{xx}^{-1}k_y^2\right)\Big]D_x+\\
&+\Big[-\mu_{yx}^{-1}\left(\varepsilon_{zy}^{-1}k_xk_y-\varepsilon_{yy}^{-1}k_xk_z\right)-\mu_{yy}^{-1}\left(\varepsilon_{xy}^{-1}k_xk_z-\right.\\
&\left.-\varepsilon_{zy}^{-1}k_x^2\right)-\mu_{yz}^{-1}\left(\varepsilon_{yy}^{-1}k_x^2-\varepsilon_{xy}^{-1}k_xk_y\right)+\\
&+\mu_{xx}^{-1}\left(\varepsilon_{zy}^{-1}k_y^2-\varepsilon_{yy}^{-1}k_yk_z\right)+\mu_{xy}^{-1}\left(\varepsilon_{xy}^{-1}k_yk_z-\right.\\
&\left.-\varepsilon_{zy}^{-1}k_xk_y\right)+\mu_{xz}^{-1}\left(\varepsilon_{yy}^{-1}k_xk_y-\varepsilon_{xy}^{-1}k_y^2\right)\Big]D_y+\\
&+\Big[-\mu_{yx}^{-1}\left(\varepsilon_{zz}^{-1}k_xk_y-\varepsilon_{yz}^{-1}k_xk_z\right)-\mu_{yy}^{-1}\left(\varepsilon_{xz}^{-1}k_xk_z-\right.\\
&\left.-\varepsilon_{zz}^{-1}k_x^2\right)-\mu_{yz}^{-1}\left(\varepsilon_{yz}^{-1}k_x^2-\varepsilon_{xz}^{-1}k_xk_y\right)+\\
&+\mu_{xx}^{-1}\left(\varepsilon_{zz}^{-1}k_y^2-\varepsilon_{yz}^{-1}k_zk_y\right)+\mu_{xy}^{-1}\left(\varepsilon_{xz}^{-1}k_yk_z-\right.\\
&\left.-\varepsilon_{zz}^{-1}k_xk_y\right)+\mu_{xz}^{-1}\left(\varepsilon_{yz}^{-1}k_xk_y-\varepsilon_{xz}^{-1}k_y^2\right)-\\
&-\varepsilon_0\mu_0\omega^2\Big]D_z=0. \qquad (16)
\end{aligned}
$$

It is obvious that in general, this system of equations (14)–(16) is somewhat redundant in terms of a number of elements. Nevertheless, this system (14)–(16) is extremely attractive in the sense that it allows one to obtain dispersion relations for any directions of a wave vector at any linear polarization. This means that using this system in a future it will be possible to construct a topology of a wave vectors surface in explicit form. In addition, it is possible, as will be shown below, to obtain the Snell relations, which relate the angles of incidence and refraction for a wave vector for mentioned type linear polarization.

From the point of view of the generality of the representation, it was important namely to identify the initial principles of the genesis of properties in the formalism of optical axes both electric and magnetic nature at arbitrary orientation of the intrinsic tensors ε and μ both before and after their diagonalization in relation to the external coordinate system in the reflection/refraction problem. The analysis provided an understanding of the situation and the requirement for the crystal that at least the main cross section of one of the tensors be in the plane of incidence. This can be used in reflectance spectrophotometry when solving the inverse problem of identifying the anisotropy parameters of complex media of the indicated type.

When specifying a wave vector in one of the coordinate planes, in accepted case it is a *xy* plane, as well as a direction of an electric field induction vector, a number of elements in a system (14)–(16) is significantly reduced. Thus, by choosing a direction of an electric induction vector and postulating the directions of a wave vector in its projections onto selected axes, it is easy to obtain the dispersion law in a general form of the tensors $\hat{\varepsilon}$ and $\hat{\mu}$.

The mentioned dispersion law is understood here as a form of a wave vector as a function of frequency, parameters of tensor coefficients and an angle of propagation, or more precisely, refraction. In fact, when specifying the components of a induction vector **D** it is determinant of system (16)–(18) that represents a dispersion law.

For example, for a simplest case of a linearly polarized wave in a medium being isotropic in optical properties, system (14)–(16) degenerates to determinant (17), which leads to well-known dispersion equation (18) for two types of polarization both for a plane of incidence and normal to it

$$
\begin{vmatrix}
\mu_{zz}^{-1}\varepsilon_{zz}^{-1}k_y^2- & -\mu_{zz}^{-1}\varepsilon_{yy}^{-1}k_xk_y & 0\\
-\varepsilon_0\mu_0\omega^2 & & \\
-\mu_{zz}^{-1}\varepsilon_{xx}^{-1}k_xk_y & \mu_{zz}^{-1}\varepsilon_{yy}^{-1}k_x^2- & 0\\
 & -\varepsilon_0\mu_0\omega^2 & \\
0 & 0 & \mu_{yy}^{-1}\varepsilon_{zz}^{-1}k_x^2+\\
 & & +\mu_{xx}^{-1}\varepsilon_{zz}^{-1}k_y^2-\\
 & & -\varepsilon_0\mu_0\omega^2
\end{vmatrix}=0, \qquad (17)
$$

$$
k_x^2+k_y^2=\frac{\varepsilon_0\mu_0\omega^2}{\varepsilon_{ii}^{-1}\mu_{ii}^{-1}}. \qquad (18)
$$

Indeed, for an isotropic medium, the tensors degenerate to a spherical form, turn into scalars, and the dispersion law corresponds to a well-known expression where all diagonal components of tensors $\hat{\varepsilon}$ and $\hat{\mu}$ are the same and all off diagonal components are zero (18).

Essentially, a carried out limiting verification of system (14)–(16) indicates the correctness of its representation. This gives grounds to use these equations to represent a dispersion law also for anisotropy of the properties of the medium. Accordingly, for a chosen anisotropy of a most general type for a polarization of a vector **D** being normal to a plane of incidence (*xy*), the dispersion relation corresponds to expression (19)

$$
\begin{aligned}
&k^{-2}\varepsilon_0\mu_0\omega^2=\varepsilon_{zz}^{-1}\left(\mu_{xx}^{-1}\cos^2\psi+\mu_{yy}^{-1}\sin^2\psi\right)+\\
&+\varepsilon_{zz}^{-1}\left(\mu_{xy}^{-1}+\mu_{yx}^{-1}\right)\sin\psi\cos\psi-\\
&-\mu_{yz}^{-1}\left(\varepsilon_{yz}^{-1}\sin^2\psi+\varepsilon_{xz}^{-1}\sin\psi\cos\psi\right)-\\
&-\mu_{xz}^{-1}\left(\varepsilon_{xz}^{-1}\cos^2\psi+\varepsilon_{yz}^{-1}\sin\psi\cos\psi\right). \qquad (19)
\end{aligned}
$$

For a wave polarized in a plane of incidence (*xy*), the dispersion relation corresponds to expression (20)

$$
\begin{aligned}
&k^{-2}\varepsilon_0\mu_0\omega^2=\mu_{zz}^{-1}\left(\varepsilon_{xx}^{-1}\cos^2\psi+\varepsilon_{yy}^{-1}\sin^2\psi\right)+\\
&+\mu_{zz}^{-1}\left(\varepsilon_{xy}^{-1}+\varepsilon_{yx}^{-1}\right)\sin\psi\cos\psi-\\
&-\varepsilon_{zy}^{-1}\left(\mu_{zy}^{-1}\sin^2\psi+\mu_{zx}^{-1}\cos\psi\sin\psi\right)-\\
&-\varepsilon_{zx}^{-1}\left(\mu_{zx}^{-1}\cos^2\psi+\mu_{zy}^{-1}\sin\psi\cos\psi\right). \qquad (20)
\end{aligned}
$$

In both cases considered (19), (20), a refractive index is a function of a propagation angle ψ. In terms of generally accepted expressions, this corresponds to condition that both waves are unusual when they propagate in a *xy* plane.

It should be noted that the dispersion law and a condition of continuity of the tangential components of a wave vectors at an interface make it possible to write relations for Snell's law and express an angle of incidence as a function of the angle of refraction. For example, for polarization of a wave in a *xy* plane, expression (21) represents a generalized Snell's law.

$$\varphi(\psi) = \arcsin\Big\{\Big[\mu_{zz}^{-1}\left(\varepsilon_{xx}^{-1}\cos^2\psi + \varepsilon_{yy}^{-1}\sin^2\psi\right) + \\ +\mu_{zz}^{-1}\left(\varepsilon_{xy}^{-1} + \varepsilon_{yx}^{-1}\right)\sin\psi\cos\psi - \\ -\varepsilon_{zy}^{-1}\left(\mu_{zy}^{-1}\sin^2\psi + \mu_{zx}^{-1}\sin\psi\cos\psi\right) - \\ -\varepsilon_{zx}^{-1}\left(\mu_{zx}^{-1}\cos^2\psi + \mu_{zy}^{-1}\sin\psi\cos\psi\right)\Big]^{-0.5}\frac{\sin\psi}{\varepsilon_1^{0.5}\mu_1^{0.5}}\Big\}. \quad (21)$$

Here ε_1 and μ_1 are electric permittivity and magnetic permeability of upper isotropic medium (for example, of vacuum).

Figure 2 represents the dependence of an angle of refraction ψ as a function of an angle of incidence φ (21) for a wave polarized in a plane of incidence *xy*. As mentioned relationship (21) was obtained on a basis of a dispersion law (20) and the condition of continuity of normal components of an electric induction vector at the interface. In essence, relation (21) in Fig. 2 reflects the Snell law of refraction for a vector **k** for some waves types including that of an usual wave (is indicated as *1*) and others (extraordinary) for wave entrance a medium of mentioned types at some concrete meanings of components of electric and magnetic tensor.

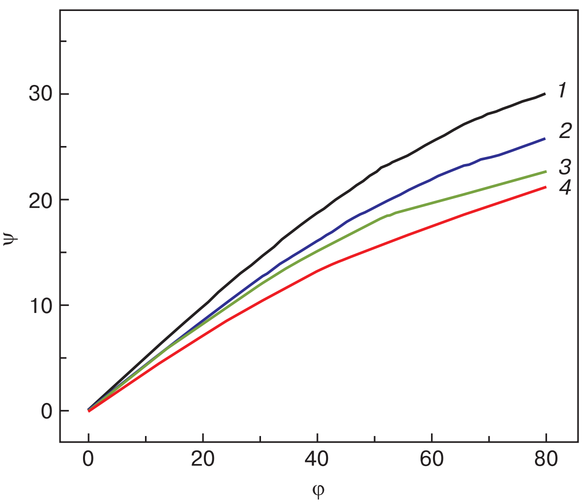

Figure 2. An example of Snell's refraction law for a usual wave and an extraordinary wave being polarized by vector **D** in a plane of incidence (*xy*) for certain values of a components of an electrical and magnetic tensors: (*1*) $\varepsilon_{xx} = \varepsilon_{yy} = \varepsilon_{zz} = 4$; $\mu_{xx} = \mu_{yy} = \mu_{zz} = 1$; $\varepsilon_{ik} = 0$; $\mu_{ik} = 0$; (*2*) $\varepsilon_{xx} = \varepsilon_{yy} = \varepsilon_{zz} = 4$; $\mu_{xx} = \mu_{yy} = \mu_{zz} = 2$; $\varepsilon_{ik} = 0$; $\mu_{ik} = 1$; (*3*) $\varepsilon_{xx} = \varepsilon_{yy} = \varepsilon_{zz} = 4$; $\mu_{xx} = \mu_{yy} = \mu_{zz} = 2$; $\varepsilon_{ik} = 1$; $\mu_{ik} = 1$; (*4*) $\varepsilon_{xx} = \varepsilon_{yy} = \varepsilon_{zz} = 4$; $\mu_{xx} = \mu_{yy} = \mu_{zz} = 2$; $\varepsilon_{ik} = 1$; $\mu_{ik} = 0$

As seen (see Fig. 2) a line *1* corresponds to a mentioned usual isotropic medium, line *2* – corresponds to medium being isotropic in electrical properties, but anisotropic in magnetic ordering, line *3* – belongs to a medium being double anisotropic, line *4* – represent electrically anisotropic medium at magnetic isotropy. As follows from Fig. 2, the wave with *xy*-polarization is most strongly affected by anisotropy of an electrical nature. This wave is refracted at an interface more strongly; magnetic anisotropy has a relatively weaker effect on a refractive index. Respectively, it can be argued that when the wave is polarized normally to a plane of incidence (*z*-polarization), a dispersion law is dominated by magnetic anisotropy.

Unfortunately, the relationships between the angles of incidence and refraction do not allow us to specify the details of a spatial dependence of a wave vector. But expressions (19) and (20) themselves make it possible to graphically display a topology of a wave vector in a plane of incidence. In other words, (19) and (20) allow to construct a section of a surface of wave vectors in one of a coordinate planes (of incidence-reflection-refraction) and not in a system of the main axes of tensors as it was discussed in the works of Fresnel. Here we are talking about a coordinate system corresponding to an external problem of refraction/reflection at an interface. In essence, here a wave vector, expressed only through the components of these tensors determines a topology of a spatial distribution of an effective refractive index in a plane of incidence for a wave of corresponding type of polarization. Respectively a section of a ray surface can be realized according to a well-known procedure with a section of the wave surface by replacing in expression for $k(\psi)$ the components of the tensors ε_{ik}^{-1} and μ_{ik}^{-1} onto their inverses. That is, by representing of $k(\psi)$ as a function of a component of ε_{ik}, μ_{ik} instead of ε_{ik}^{-1}, μ_{ik}^{-1} there have been got a section of a ray surface $s(\psi)$.

Actually, according to the general concepts, the wave vector in the analysis is displayed as a function of the inverse components of the tensors ε_{ik}^{-1}, μ_{ik}^{-1} and not as ε_{ik}, μ_{ik}. When constructing the section of the ray vector **s**, generalized principles of correlating the wave and ray surfaces through the direct and inverse tensors of both dielectric and magnetic permeability were applied. That is, the resulting expression for the wave vector is chosen as the base one, and the expression for the ray vector is constructed on its basis by involving the inverse tensors/ their components to those already used.

Both types of sections, that is, $k(\psi)$ and $s(\psi)$, are presented in Fig. 3 for one particular type of anisotropy of a medium during a propagation of two types of waves, namely being polarized in a plane of incidence (*xy*-polarization) and normal to *xy* (*z*-polarization). Essentially, Fig. 3 shows the cross-section of a wave surface as a topology of an effective refractive index in a plane of incidence under a designation k_{xy} and k_z respectively for a *xy*-polarization and *z*-polarization. For ray surface there accepted indications as a s_{xy} and s_z.

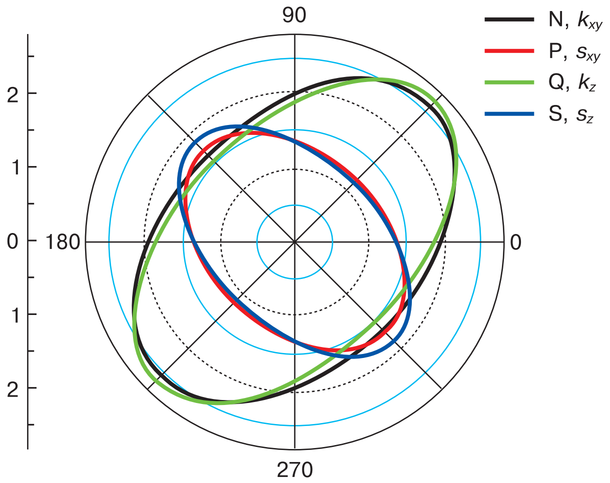

Figure 3. A type of effective cross section in terms of an effective refractive index for a wave surface k and for a ray surface) in a plane of incidence for a transverse wave being polarized in a plane of incidence k_{xy}, s_{xy} and normal to the plane of incidence k_z, s_z. Here during ray surface construction its real meaning was multiplied by 3. The crystalline medium has anisotropy of the form $\varepsilon_{xx} = \varepsilon_{yy} = \varepsilon_{zz} = 5$; $\varepsilon_{xy} = \varepsilon_{yx} = 2$; $\varepsilon_{xz} = \varepsilon_{zy} = \varepsilon_{zx} = \varepsilon_{yz} = 1$; $\mu_{xx} = \mu_{yy} = \mu_{zz} = 1$; $\mu_{xy} = \mu_{yx} = 0.5$; $\mu_{xz} = \mu_{zy} = \mu_{zx} = \mu_{yz} = 0.2$

As can be seen (see Fig. 3), both types of a polarization in a medium with a selected type of anisotropy (see Fig. 3) correspond to an extraordinary wave since an effective refractive index in a representation of a wave/ray vector is not isotropic for a plane xy. In short, in a xy plane, two directions are observed along which the phase and ray velocities for both types of polarization are the same. May be these directions correspond to so called bi-normals for a wave vector and to bi-radials for a ray vector. It is not clear however whether the directions of intersection of a black and green contours are true bi-normals or are they only projections of these lines onto a plane of incidence xy.

This question can be answered after bringing the tensors from Fig. 3 to main axes (22)

$$\hat{\varepsilon} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 6+\sqrt{3} & 0 \\ 0 & 0 & 6-\sqrt{3} \end{pmatrix},$$

$$\hat{\mu} = \begin{pmatrix} 0.5 & 0 & 0 \\ 0 & 1.25+0.5\sqrt{0.57} & 0 \\ 0 & 0 & 1.25-0.5\sqrt{0.57} \end{pmatrix}. \quad (22)$$

In this case, the orientation of unit vectors of the new main axes X, Y, Z for the tensor $\hat{\varepsilon}$ meets to relation

$$\begin{array}{l} 1:-1:\ \ 0 \\ 1:\ \ 1:-1-\sqrt{3} \\ 1:\ \ 1:-1+\sqrt{3} \end{array} \quad (23)$$

For the reduced tensor $\hat{\mu}$ the orientation of unit vectors of the new principal axes X_m, Y_m, Z_m corresponds to relation (24)

$$\begin{array}{l} 1:-1:\ \ 0 \\ 1:\ \ 1:-1.25-2.5\sqrt{0.57} \\ 1:\ \ 1:-1.25+2.5\sqrt{0.57} \end{array} \quad (24)$$

Accordingly, without any geometric constructions, it is clear that a main sections in a system of reduced axes X, Y, Z and X_m, Y_m, Z_m for both a tensor $\hat{\varepsilon}$ and a tensor $\hat{\mu}$ respectively do not coincide with a original coordinate system x, y, z of accepted external problem. Nevertheless in this case, one of a main axes for the tensors $\hat{\varepsilon}$ and $\hat{\mu}$ is located in a plane of incidence of external system.

Before nowadays, in classical works only an electrically anisotropic optical medium was discussed. After reducing its tensor $\hat{\varepsilon}$ to the main axes, this tensor has two or three unequal values of diagonal components. The last case means that medium is bi-axial. When a beam propagates in a main plane and is polarized normal to a plane of this main section an effective refractive index does not depend on an angle (this means an ordinary wave type). For a wave polarized in a plane of a main section an effective refractive index depends on a refractive angle, that means an extraordinary wave. In classical works the directions called by bi-normals (velocities of both waves coincide) are represented only in a main sections of the XY or XZ or YZ type.

Based on this analysis it is clear that a coincidence of phase velocities along a direction of a bi-normals type is realized for polarization not only for main planes, but also outside them. This exactly correlates with a concept of conical refraction, which will be discussed below.

Once more an example of double anisotropy represents a phenomenon of conical refraction more clearly in a following Fig. 4, in which the values of the magnetic tensor components of accepted hypothetical medium under consideration are some changed in comparison with the parameters of Fig. 3.

$$\begin{array}{l} \varepsilon_{xx} = \varepsilon_{yy} = \varepsilon_{zz} = 5;\ \ \varepsilon_{xy} = \varepsilon_{yx} = 2; \\ \varepsilon_{xz} = \varepsilon_{zy} = \varepsilon_{zx} = \varepsilon_{yz} = 1;\ \ \mu_{xx} = \mu_{yy} = \mu_{zz} = 2; \\ \mu_{xy} = \mu_{yx} = 0.25;\ \ \mu_{xz} = \mu_{zy} = \mu_{zx} = \mu_{yz} = 0.1. \end{array} \quad (25)$$

Here as mentioned an electric tensor is the same as before (see Fig. 3) but magnetic tensor components are changed and, respectively, the directions both of bi-normals and of bi-radials are seen more clearly. That is a magnetic order variation acts onto both types of polarization.

Such in Fig. 4 two directions of bi-normals correspond to lines from a center to the points of self-intersection of black and green contours being the sections of wave surfaces for two polarizations. Accordingly, the directions

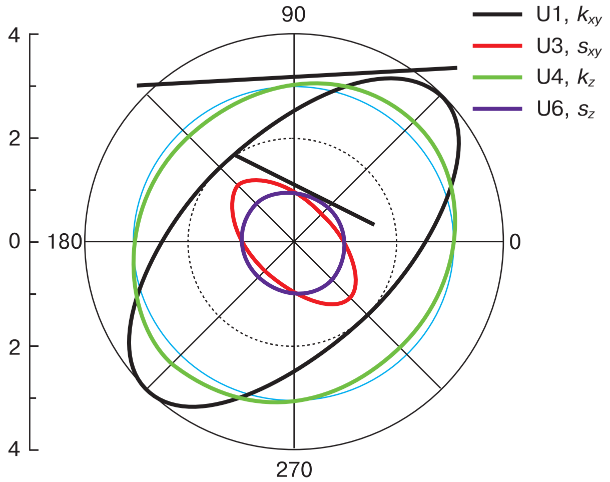

Figure 4. The type of effective cross section in terms of an effective refractive index for a wave surface *k* and ray surface *s* in a plane of incidence for a transverse wave being polarized in a plane of incidence k_{xy}, s_{xy} and normally to a plane of incidence k_z, s_z. A line that touches a beam surface (red and violet contours) determines the direction of beam splitting during internal conical refraction. The line that touches the wave surface (black and green contours) determines the directions of the incoming connecting rays under external conical refraction. Here during ray surface construction its real meaning was multiplied by 3. The crystalline medium has double anisotropy in accordance with (25)

of bi-radials correspond to axes of Cartesian system (the points of self-intersection of a ray surface – red and violet contours – lie along the *x* and *y* axes). In accordance with a general canons, for a ray entering from an isotropic medium to crystal along bi-normals being oriented normally to sample surface (from an origin of coordinates to a point of intersection of a black and green contours), there will be realized so called internal conical refraction. That is, in a medium a beam will split and propagate in a direction of the points of contact of the sections of a ray surface with a common plane being normal to an initial direction of a wave vector. Here it makes sense to talk more about a longitudinal section of a cone of internal refraction. That is, a beam containing both types of linear polarization at its normal incidence will split along a longitudinal section of a cone. According to Snell's law, there will be no refraction for direction along the bi-normal axis for *z*-polarization due to normal incidence, but a ray corresponding to a *xy*-polarization will go along a generatrix of a cone towards the red contour.

On the other hand, a beam propagating in isotropic medium at a small angle to an *y*-axis, which after refraction goes along an *y*-axis, may also have a component directed toward a center from an area where a common plane touches the wave surface (the black and green contours) before entering the crystal. After merging into a single beam in a medium and propagating through a crystal, this beam will again split into two beams. Here one of beams will go near a direction of a *y*-axis and another at an angle to a central beam in a direction of a plane touching a wave surface, green and black contours. In this version, a phenomenon of an external conical refraction is realized, but the figure again shows not cone entirely, but their longitudinal sections by a *xy* plane. Finally: the beam entering normally a crystalline medium along a line of intersection of green and black contours, that is, along a bi-normal, has a cross-section of a cone of internal refraction determined by the points of tangency of the plane perpendicular to bi-normals on a ray surface (red and violet contours). Rays entering a crystal along an *y* axis and along a generatrix of a cone towards a center from a points of contact of a wave surface (black and green contours) with a plane being perpendicular to a bi-radial will be combined into one, followed by splitting in outside.

Further it is a reason to show a view of a section for wave vector to the power 2 in terms effective refractive index. So Fig. 5 shows a wave vector to the power 2 for two waves polarized in a plane of incidence and normally to it when these entering a medium being an anisotropic due to both an electrical and magnetic ordering. In fact, Fig. 5 shows the degree of influence of a scale of diagonal components on a type of a wave surface for two characteristic wave polarizations. Here, both waves under a numerical equality of all non-diagonal electric and magnetic tensor components are extraordinary these. In this approximation as seen the bi-normals are directed along the *x* and *y* axes.

A stronger elongation of a wave section for *z*-polarization indicated that for this type wave a dispersion law is dominated by magnetic ordering, even though the numerical values of all non-diagonal components for both electric and magnetic tensors are equal to each other.

It is important that if a coordinate axes are chosen so that components of type ε_{iz} and ε_{zi} are equal to zero, and a form of the tensor $\hat{\mu}$ is general, relation (19) simplifies

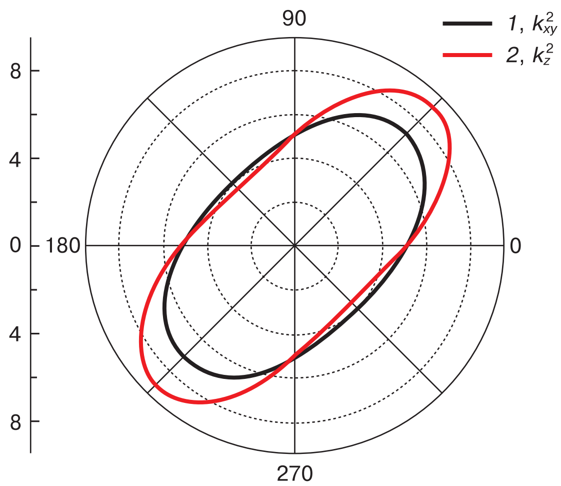

Figure 5. The topology of a cross section for a wave vector to the power 2. A square of effective refractive index for waves entering a double anisotropic optical medium when these are polarized in a plane of incidence (*1*) and normally to it (*2*). The crystalline medium parameters are: $\varepsilon_{xx} = \varepsilon_{yy} = \varepsilon_{zz} = 4$; $\mu_{xx} = \mu_{yy} = \mu_{zz} = 2$; $\varepsilon_{ik} = 1$; $\mu_{ik} = 1$

to form (26), in which a refractive index and wave vector are determined only by magnetic anisotropy through the components μ_{xy}, μ_{yx}. However, a wave of such z-polarization (an electric field induction vector is normal to xy plane) still remains extraordinary (26)

$$k^2 = \frac{\varepsilon_0\mu_0\omega^2}{\varepsilon_{zz}^{-1}\left(\mu_{xx}^{-1}\cos^2\psi + \mu_{yy}^{-1}\sin^2\psi\right) + \varepsilon_{zz}^{-1}\left(\mu_{xy}^{-1} + \mu_{yx}^{-1}\right)\sin\psi\cos\psi}. \quad (26)$$

In a more particular case, for example, if all diagonal components in (26) are equal and if this tensor is symmetric, which initially should be assumed if there is no external influence, it is obvious that an expression in question (26) can be reduced to the form (27). This means that as before in a plane of incidence (xy) there are two mutually perpendicular directions, named bi-normals, between which an effective refractive index changes from a minimum value to a maximum in accordance with dispersion relation (27)

$$k^2 = \frac{\varepsilon_0\mu_0\omega^2}{\varepsilon_{zz}^{-1}\left(\mu_{xx}^{-1} + \mu_{xy}^{-1}\sin 2\psi\right)}. \quad (27)$$

Naturally, if the tensor $\hat{\mu}$ is antisymmetric in the components μ_{xy} and μ_{yx}, for example, under the influence of an external magnetic field applied along a z-axis, as well as when there is no magnetic anisotropy at all, the dependence of a refractive index on the angle ψ disappears. Respectively, in such type of polarization (along z), the dispersion law corresponds to an ordinary wave

$$k^2 = \frac{\varepsilon_0\mu_0\omega^2}{\varepsilon_{zz}^{-1}\mu_{xx}^{-1}}. \quad (28)$$

For a wave being polarized in an incidence plane xy, a similar analysis can be performed, which will show the conditions for a components of a dielectric tensor, which will lead to isotropy of the wave vector. As before, we keep in mind that polarization as a state of the electric vector is set by the position of the induction vector **D**.

Next, it makes sense to consider once more particular case of anisotropy within an accepted coordinate system (x, y, z). In this approximation, one of a two waves being polarized normally to a plane of incidence (xy) is ordinary, and another that is, accordingly, extraordinary. Applying a formulated dispersion laws it is easy to see that for anisotropy of a type (29)

$$\hat{\varepsilon} = \begin{pmatrix} \varepsilon_{xx} & \varepsilon_{xy} & 0 \\ \varepsilon_{yx} & \varepsilon_{yy} & 0 \\ 0 & 0 & \varepsilon_{zz} \end{pmatrix}, \quad \hat{\mu} = \begin{pmatrix} \mu_{xx} & 0 & \mu_{xz} \\ 0 & \mu_{yy} & \mu_{yz} \\ \mu_{zx} & \mu_{zy} & \mu_{zz} \end{pmatrix}, \quad (29)$$

a wave being z-polarized becomes ordinary that of a type (28), and a wave being xy-polarized will be extraordinary, corresponding to a dispersion law (30)

$$k^2 = \frac{\varepsilon_0\mu_0\omega^2}{\mu_{zz}^{-1}\left(\varepsilon_{xx}^{-1} + \varepsilon_{xy}^{-1}\sin 2\psi\right)}. \quad (30)$$

The form of dispersion law (30) corresponds to a more rational orientation of the crystalline axes and reflective surface of a medium in relation to the coordinate system of tensor coefficients that is to more benefit sample surface orientation in an external problem of an incidence-reflection. The rationality of this representation means that with a dielectric tensor of a form (29), a wave vector of the incident and refracted waves are placed in a main plane of tensor (XY). The number of optical axes as usual is determined by a number of unequal diagonal elements of tensor (29) after its reduction to main axes. Respectively the optical axes that is bi-normales as directions with the same values of a phase velocity for both types of polarization are oriented precisely in a plane of incidence (xy) because this plane coincides the maim plane of tensor. So for an approximation (29) the medium can be characterized by either one or two optical axes of only electrical nature, which follows from reducing the tensor $\hat{\varepsilon}$ (29) to principal axes.

For example for an anisotropy of a type (31) a dielectric constant tensor after a reducing it to the main axes corresponds to the expression (32).

$$\begin{gathered} \varepsilon_{xx} = \varepsilon_{yy} = \varepsilon_{zz} = 4;\ \ \varepsilon_{xy} = \varepsilon_{yx} = 2; \\ \varepsilon_{xz} = \varepsilon_{zy} = \varepsilon_{zx} = \varepsilon_{yz} = 0;\ \ \mu_{xx} = \mu_{yy} = \mu_{zz} = 2; \\ \mu_{xy} = \mu_{yx} = 0;\ \mu_{ik} - \text{any}. \end{gathered} \quad (31)$$

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 4 \end{pmatrix} \quad (32)$$

Respectively, the orientation of the unit vectors of new main axes X, Y, Z for the tensor $\hat{\varepsilon}$ (31) meets the relation (33)

$$\begin{matrix} X & 1: & 1: & 0 \\ Y & 1: & -1: & 0 \\ Z & 0: & 0: & 1 \end{matrix} \quad (33)$$

The relation (33) indicates that choosing a rational tensor coordinate system that coincides with a coordinate system of an external reflection problem one can represent a section of a wave surface in an incidence plane (xy). Respectively it is possible to indicate the directions of bi-normals and bi-radials being placed in xy plane too. As before here optical axes determine the cases of usual refraction or a double refraction for both types of wave polarization. So for example for tensor (29) of the form (31) a wave vector and a ray vector as the effective values of a refractive index after reduction to the main axes correspond to a Fig. 6.

Following Fig. 6 at accepted anisotropy type there are two optical axes in medium. Namely for a wave being xy-polarized there is an extraordinary wave, whereas a z-polarized wave is ordinary. It should be mentioned that if a tensor component ε_{zz} would be for example equal of

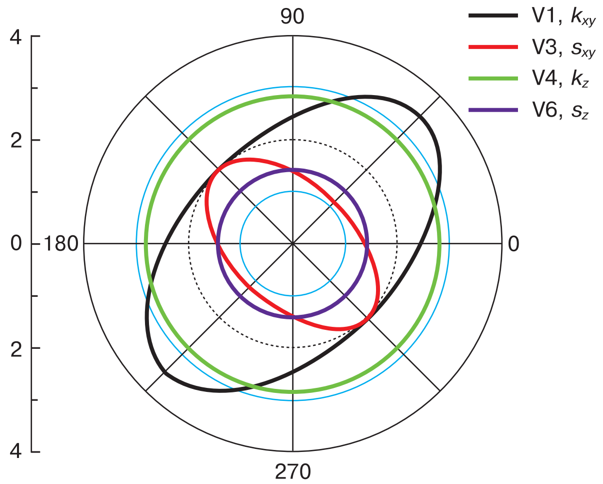

Figure 6. The section of a wave surface and of a ray surface in plane of incidence/reflection (*xy*) for ordinary (contours green, violet) and extraordinary (contours black, red) waves via a refractive angle for a one particular type of anisotropy (31). As before k_{xy} and k_z are wave vectors, s_{xy} and s_z are ray vectors. The elongated figures – for an extraordinary wave (30), a circle – for an ordinary wave (28). During construction of s_{xy} and s_z contours a real meanings were multiply by three

2 or of 6 the picture of refraction would correspond to a medium being one-axes that. Computer graphics tools make it possible to represent physical characteristics in a three-dimensional image. Accordingly, one of the fragments for a wave surface at a *xy*-polarization for an anisotropy medium of the form (31) is displayed on Fig. 7. In general, the overall outline of Fig. 7 repeats the black outline in Fig. 6, which denotes the cross section of a wave vector k_{xy} for an extraordinary wave being polarized in the plane (*xy*).

Note that as mentioned above in a general case a direction of an energy flux density vector **s** does not coincide

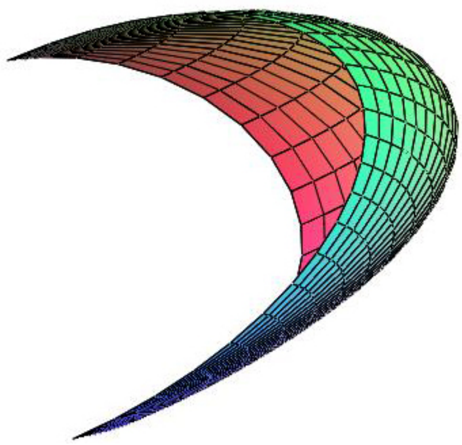

Figure 7. View of one part of a wave surface for a wave being linear polarized in a plane of incidence. An electrically anisotropic medium has a parameters of a type (31), a wave corresponds to an extraordinary that. Range variation of angles is: a refraction (latitude) angle, ψ is: $0 - \pi$ (placed in a plane of Fig. 7), azimuth (longitude) angle φ is: $0 - \pi/2$ (is oriented normally to a Fig. 7 plane)

with the direction of a wave vector **k**. Under an anisotropy of type (29) both parameters lie in the same plane *xy*, since the direction cosine between the *z* and *Z* axes is equal to unity. Respectively the angle *u* between vectors **k** and **s** is defined as (34) (this thesis is presented at Fig. 1). The phase velocity υ and group velocity *q* are related through the expression $\upsilon = q \cos u$ and an angle u is the functions of electric tensor components (31).

$$\cos u = \frac{\varepsilon_{xx}^{-1} + \varepsilon_{xy}^{-1} \sin 2\psi}{\left(\varepsilon_{xx}^{-12} + 2\varepsilon_{xx}^{-1}\varepsilon_{xy}^{-1} \sin 2\psi + \varepsilon_{xy}^{-12}\right)^{0.5}}. \tag{34}$$

Thus, for the case of a particular type of dispersion law of type (29), a bi-anisotropic medium can be represented in terms of familiar concepts. In the accepted approximation, a plane of a main section coincides with the *xy* plane. In *xy* plane either two or one optical axes are located. A wave being polarized according to a vector of electrical induction in a plane of mentioned main section (*xy*) is extraordinary, and a wave being polarized perpendicular to this plane is ordinary that. For an extraordinary wave, either one or two special directions are possible.

The formulated dispersion relations make it possible to write Snell's law for the wave vector and, in a future, to obtain a Fresnel ratio for amplitudes for electric intensity vectors, namely a ratio of a reflected wave amplitude to an incident that. Relationships for amplitudes are available for implementation based on standard continuity conditions.

Returning to the most general form of double-type anisotropy, we can conclude that an arbitrary choice of a common coordinate system for tensors means a more complicated presentation of two extraordinary waves with polarization in a plane of incidence and perpendicular to it. The graphical construction of a topology of a wave surface section in a *xy* plane displays the projections of special directions onto the plane of incidence. The special directions themselves in such general case are placed out of a plane of incidence. For a more rational choice of an experiment geometry a plane of incidence a wave vector must be coincided with one of a main planes of an electric or magnetic permeability tensors that is with plane of crystallography symmetry. At this case there is a hope that optic axes will be entirely placed in the plane of incidence. This means that some of off-diagonal elements of the permeability tensors may get equal to zero. If at least one pair of non-diagonal elements is equal to zero, the tensor is quite simply reduced to the main axes.

4. Conclusion

The Maxwell's equations in presentation of a coordinate system of an external problem of a reflection/refraction for a transverse electric-magnetic wave allows anybody to write down a wave dispersion law after refraction of a polarized wave into an anisotropic medium being ordered by both electrical and magnetic properties. A double type of ordering it makes sense to represent in a components of a polar tensors for electric and magnetic permeability taking into account a fact that these tensors must be symmetrical at the absence of medium own optical activity or of external influence.

Tensor equations for a wave vector being presented in coordinate form allow to obtain a dispersion equations for a wave vector and a ray vector at an interface in a plane of incidence being that of accepted tensor coordinate system as well as for other intermediate planes of incidence and to construct wave vector surface. The coupling relations for wave vectors at an interface allow by using a Snell refraction law to generalize Fresnel relations in future for reflection coefficients.

The dispersion law in a bi-anisotropic medium for a transverse wave for two types of polarization of an electric induction vector is suitable for constructing of a wave surface of a refracted wave with subsequent identification of the features of a medium by a number of optical axes, by their nature, by a type of wave refraction and by a directions of phase and group velocities.

Coordinate representation of dispersion law confirms an expediency of organizing of reflection experiment while ensuring a rational geometry of the problem. This implies an orientation of a crystalline symmetry axis of higher order into a plane of incidence. As result it make it possible to more rationally use empirical data on reflection when reconstructing the features of a crystalline medium due to the reduction in a number of tensors components during the solution of an inverse refraction/reflection problem.

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