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Adjustment of a Precise Levelling Network Without Using the Means of Line Elevations

Christina Mickrenska[1, a)], Gergana Antova[1]

[1] *University of Architecture, Civil Engineering and Geodesy, Faculty of Geodesy, 1 Hristo Smirnenski Blvd., 1164 Sofia, Bulgaria*

[a)] Corresponding author: kmikrenska_fgs@uacg.bg

Abstract. The precise geometric levelling has been the main method for establishing height reference systems since the eighteenth century. Despite evaluating the methodology of measurements, levelling equipment, and the knowledge about the systematic effect of some natural processes, some systematic errors in the levelling data processing are still used in the geodetic practice. Such a systematic data processing error uses the mean of the measured line elevations in opposite directions. A newly revealed distribution Cv(n) gives the probability that the mean of *n* observations has the least true error than each of these n observations. The maximum value of the Cv(n) probability density function is in the case of the uniformly distributed data and n=3. In this case, Cv(n=3) < 0.50. The common case for the precise levelling is a line elevation to be measured twice, i.e., n=2. Then, the probability Cv(n=2) < 0.33. Consequently, using original values of the measured line elevations as initial data in an adjustment of a levelling network is expected to give better results than their means. The expected increase in accuracy is estimated to be twice. To test this theory, we readjusted the Third Levelling of Bulgaria network /1975-1984/ by selecting one of the two measured line elevations by a greedy algorithm. Comparison of the samples of benchmark height standard errors obtained by applying only the original measurements on the one hand, and the means of the other, shows the superiority of the proposed approach. Comparison of the means by t-test confirms this conclusion at a significance level higher than 99.99%. Comparison of the variations of the benchmark standard error samples rejected the null hypothesis of the sample variance equality at a significance level higher than 99%. In addition, the analysis of the differences between the adjusted benchmark heights reveals a systematic inclination of the classical adjusted variant in the Southeast-Northwest direction greater than +0.10 mm/km. The last value is close to the estimated systematic error of the Third Levelling of Bulgaria.

Keywords: Height reference systems; jackknife; entropy; probability; precise levelling.

INTRODUCTION

Precise geometric levelling is still the most accurate geodetic method for measurement of height differences between points on the Earth’s surface. However, this method has lost a significant part of its popularity since the introduction of GNSS technologies and their tremendous evolution over the last decades. Some reasons for the decreasing popularity of the geometric levelling are related to the tediousness, time-consuming and repetitiveness of the measuring process. An additional cause is the accuracy deadlock of the method, which is due to the obsolete assumptions regarding the law of levelling error accumulation. The last reflects on the methodology of measurements, accuracy estimation, and posterior data processing. Analysing the contents of the last two specifications for precise levelling in Bulgaria [1, 2], one can see that the new specification [2] is almost the same as its predecessor [1], especially in the part of levelling data processing. Considering the adjustment of the highest order geometric levelling, there are two "indisputable" recommendations, namely:

- The mean of both elevation observations, which are measured in opposite directions, has to be used in the data processing, i.e., in the adjustment.
- The weights w of the adjusted line elevations have to be inversely proportional to the length of levelling lines L or the number of setups n along the levelling line. The first weights are usually given by equation (1). They are the preferred choice.

$$w = L^{-1} \tag{1}$$

According to some new researches [3, 4], which investigate the probability $Cv(n)$ that the mean of n observations has the least true error than each of these n observations tends to zero when $n \to \infty$. The peak of the $Cv(n)$ probability density function is in the case of the uniformly distributed data and $n=3$. In this case, $Cv(n=3) \approx 0.45$. The common case for the precise levelling is a line elevation to be measured twice, i.e., $n=2$. Then, the probability $Cv(n=2) \approx 0.33$. In the case of normally distributed data, which is a desirable situation regarding geodetic observations, we have $Cv(n=3) \approx 0.37$ and $Cv(n=2) \approx 0.30$ [3]. Thus, using the mean of both opposite measured line elevations as the initial line elevation value is a wrong decision in approximately 70% of all cases. The jeopardy of automatically applying the means as initial data in an adjustment of a levelling network is illustrated in the study [5, 8].

In addition, there are serious statistically based arguments about the relevance of weights (1) [6, 7].

So, we define our aims as follows. The main objective of this study is to show that applying the originally measured line elevations, used as initial data in an adjustment of a levelling network, leads to better results than their means. Our secondary aim is to illustrate how weights (1) could be a reason for the tilting of the levelling networks.

METHODOLOGY AND DATA

To prove our statements, we use the data from the Third Levelling of Bulgaria /1975-1984/. Figure 1 gives the network configuration. The network contains 55 levelling lines, which define 33 nodal benchmarks. Their total length is 5631.19 km. The benchmark in Varna was chosen as a datum point in our analyses.

The network is adjusted by applying two conceptually different approaches. The first approach is the classical one, which is widely used and recommended by the state specifications [1, 2]. As initial processing data, this approach uses the means of the line opposite measured elevations and the popular weights that are inversely proportional to the levelling lengths. The second approach uses these values of original measured line elevations as initial processing data that minimise the loop closing errors. The elevation weights in the second approach are assumption-free of hypotheses about the accumulation of their uncertainty, but based on the line elevation's real impact on the final network accuracy. They were calculated by following the algorithm:

1. We readjusted the Bulgarian Third Levelling network 55 times. Each time we removed a different levelling line.
2. Based on the remaining 54 lines, we adjusted the network and summed the standard errors of the adjusted benchmark heights. If some line deteriorates the network accuracy, it is normal that its skip will produce a smaller sum of the nodal benchmarks' standard errors. Thus, the less the sum of the nodal benchmarks' standard errors, the less the line weight. Since the number of levelling lines is equal for the independent adjustments, we can use the average of the nodal benchmarks' standard errors (ASE[1]) in each adjustment instead of their sums.
3. When we obtain 55 ASE, we can calculate their average ASE_{MEAN}.
4. Now we can calculate the weight w_{L_i} for each levelling line i by equation (2).

$$w_{L_i} = (ASE_i / ASE_{MEAN})^2 \tag{2}$$

[1] Average Standard Error

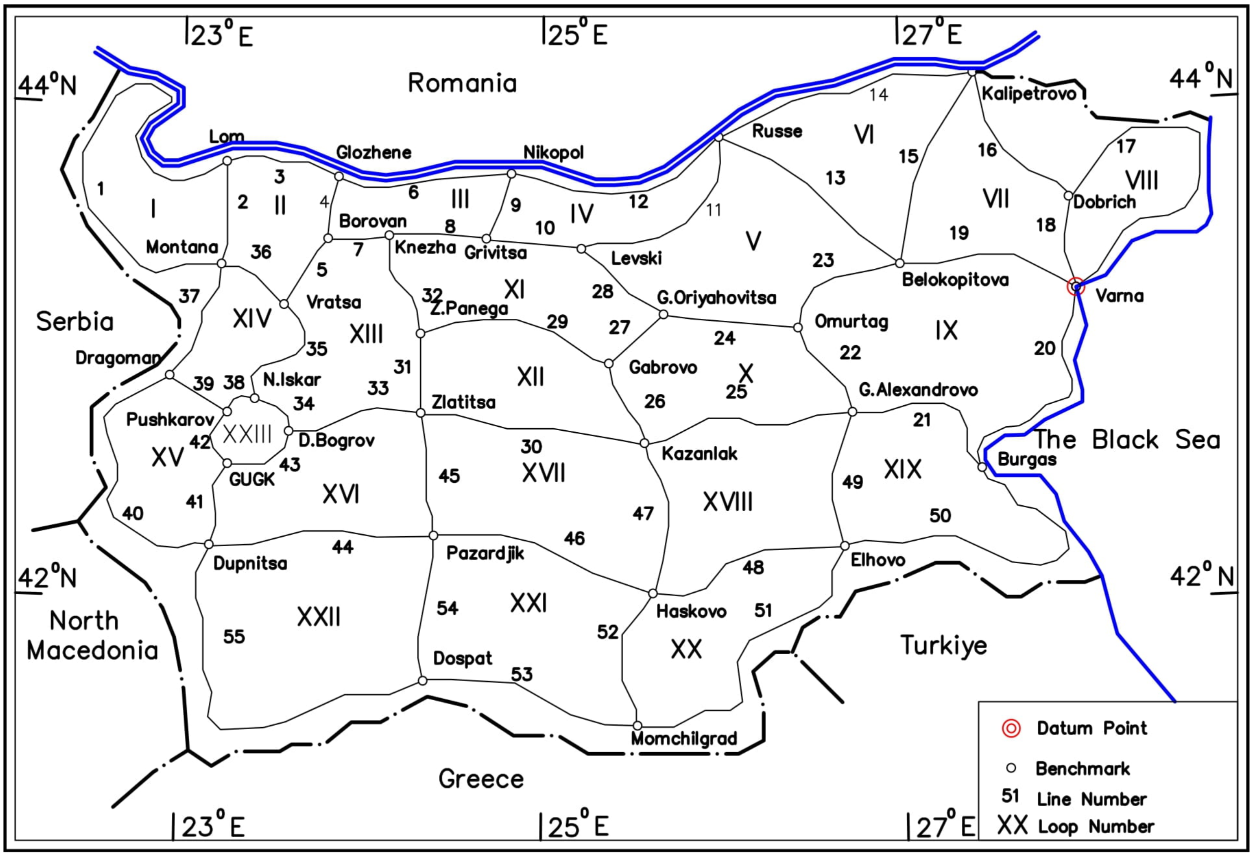

FIGURE 1. The Bulgarian Third Levelling Network.

RESULTS

Figure 2 gives the standard errors of the adjusted benchmark heights obtained from the adjustments of the Bulgaria network's third levelling by the classical approach [1, 2] and the proposed approach here. The supremacy of using original observations over the artefactual means is obvious. Results presented in Tables 1 and 2 statistically support this conclusion. Differences in the adjusted benchmark heights by both discussed approaches are illustrated in Figure 3.

TABLE 1. Results from a F-Test Two-Sample for Variances.

Description	Classical Adjustment	Proposed Adjustment
Mean, [mm]	13.646	5.920
Variance, [mm^2]	4.378	1.037
Observations	32	32
df	31	31
F	4.222	
P(T<=t) one-tail	6.5E-5	
F Critical one-tail	1.822	

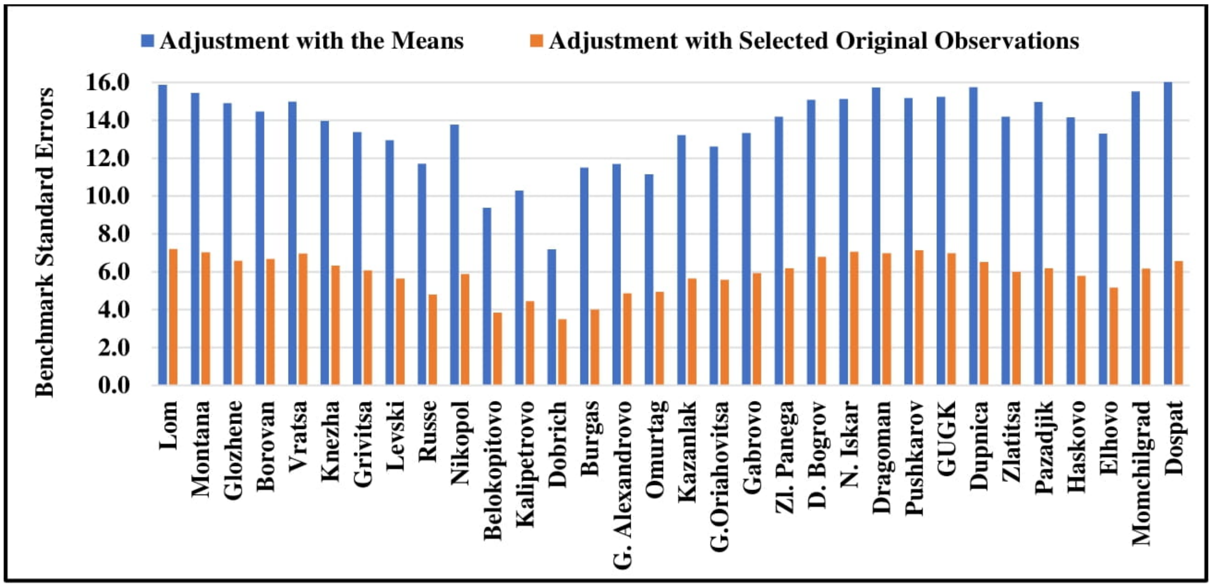

FIGURE 2. Standard errors in [mm] of the adjusted benchmark heights, obtained by the adjustment with classical initial data and weights, and the adjustment with selected original observations and Jackknife-based weights.

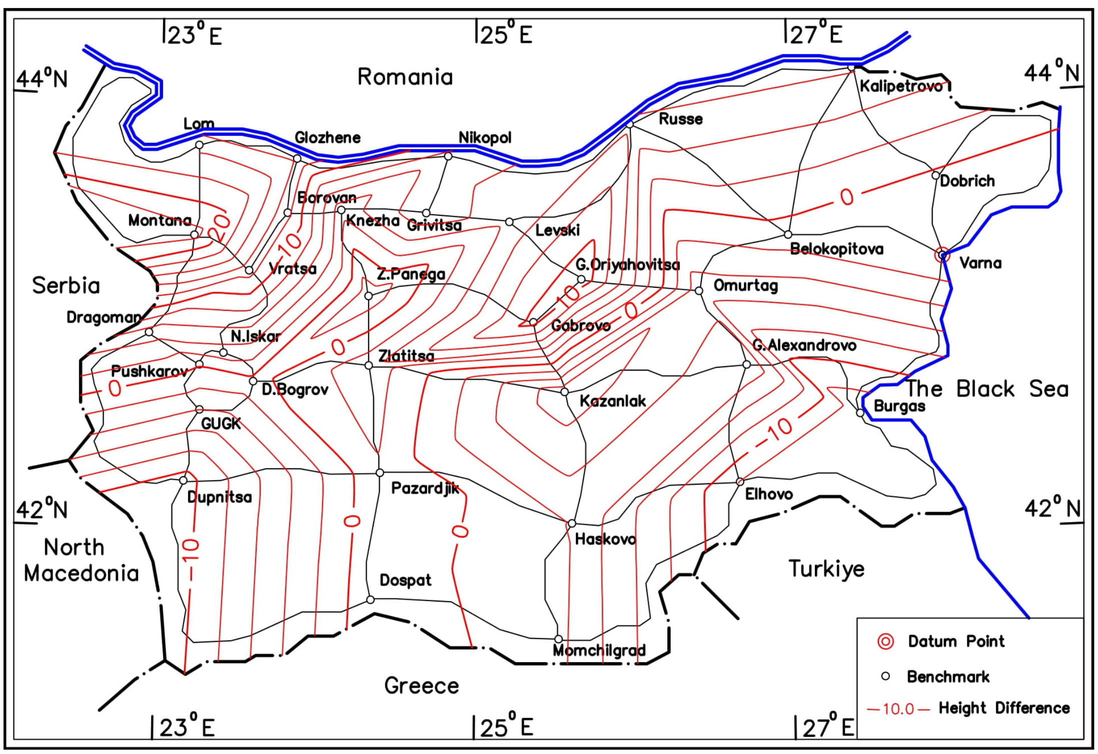

FIGURE 3. Differences in [mm] between the adjusted benchmark heights, obtained by the adjustment with classical initial data and weights, and the adjustment with selected original observations and Jackknife-based weights.

TABLE 2. Results from a t-Test: Two-Sample Assuming Unequal Variances.

Description	Classical Adjustment	Proposed Adjustment
Mean, [mm]	13.646	5.920
Variance, [mm^2]	4.378	1.037
Observations	32	32
Hypothesized Difference	0	
df	45	
t Stat	18.781	
P(T<=t) one-tail	3.2E-23	
t Critical one-tail	1.679	
P(T<=t) two-tail	6.4E-23	
t Critical two-tail	2.014	

DISCUSSION

Analysing the standard errors of the adjusted benchmark heights presented in Figure 2, one can see that the adjustment with the original observation values of line elevations is more than twice as accurate as the classical adjustment. The means of the benchmark height standard error samples of the classical and the presented variant are 13.65 and 5.92 mm, respectively. According to Table 1, which contains the comparison of these means by t-test under the assumption of unequal sample variances, the two-tailed null hypothesis H_0 ($M_1 = M_2$) is rejected at a significance level higher than 99.999%. The p-value produced by the test is $6.4*10^{-23}$, which is strong evidence for inequality of the tested means. According to Table 2, the F-test results show a higher homogeneity of the standard errors obtained by the proposed non-classical variant of adjustment. The conclusion is made at a significance level higher than 99.99%.

Are the presented results any surprise? Of course, not. There are natural consequences of the fact that the true error of the mean of two random observations of a quantity is less than 33% smaller than the true errors of the values of the two quantity observations [3, 4]. Thus, applying only the mean of a quantity of observations is not acquitted from the Cv(n) probability point of view.

Another downside of the classical adjustment variant of the precise levelling networks is the usage of the “popular” weights (1). Some recent studies [6, 7, 8] illustrate the irrelevance of the relationship between the levelling accuracy and levelling lengths. A more reasonable explanation of the accumulation of levelling uncertainty is given in the study [6], where the stochastic nature of the accumulation of a random number's sum is pointed out as the greatest accumulation error factor.

The most important results from a geodetic network adjustment are the adjusted values of initial observations. Figure 3 presents the differences between the adjusted benchmark heights, obtained by the adjustment with classical initial data and weights (1), and the adjustment with selected original observations and Jackknife-based weights. As can be seen, in the Northwestern part of the network, these differences exceed 20 mm. The positive differences are dominantly in the Northern part of the network. The negative differences are mainly in the Southern part of the network. There is no obvious correlation between the differences in the adjusted benchmark heights and the benchmark remoteness from the datum point in Varna. However, there is a remarkable tilt in the Southeast–Northwest direction. The average value of this tilt is estimated to be 0.10 mm/km. The last value is greater than the widely discussed values of the British [8] or North American [9] levelling networks.

CONCLUSION

This study compared the results from two conceptually different adjustment approaches of a precise levelling network. The first approach is the classical one, which is widely used and recommended by the state specifications [1, 2]. As initial processing data, this approach uses the means of the line opposite measured elevations and the popular weights that are inversely proportional to the levelling lengths. The second approach uses these values of original measured line elevations as initial processing data that minimize the loop closing errors. The elevation weights in the second approach are assumption-free of hypotheses about the accumulation of their uncertainty, but based on the line elevation's real impact on the final network accuracy. The comparison of the results produced by the compared approaches shows that the second one leads to more than twice as small standard errors of the adjusted benchmark heights and more homogeneous standard errors. The obtained results can be estimated as a surprise. They are a natural consequence of the probability *Cv(n)* [4, 5] that the mean of *n* random observations has the least true error among the observations. The results presented here clearly show that the prescriptions [2] for levelling data processing in Bulgaria, and not only, are obsolete, irrelevant, and not in concordance with the observed true error probabilities. Thus, a revision of the specifications [2] is needed.

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