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Integration of Teaching Higher Mathematics, Digital Technologies and Programming with Modeling Fractal Settings

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Abstract

The relevance of the presented study is due to the need to ensure the fundamental nature of education in the system of professional training of students of mathematical faculties in the field of higher mathematics, digital technologies and programming. The article discusses the features of studying the course "Geometric fractals", which is a subject that integrates knowledge in higher mathematics and digital technologies. A description of technologies for using affine and projective geometric transformations in the construction of fractals and their implementation in Maple is proposed. The laws of constructing L-systems for fractal curves on a plane using turtle graphics in Python are considered. This study is aimed at organizing the interaction of a teacher with students, where teaching aids play an important role, allowing: to increase the amount of necessary educational information; improve understanding of the studied material; to systematize knowledge in subjects; develop the research abilities of students; create individual trajectories for studying a subject; increase the use of digital technologies in teaching the discipline "Geometric fractals". To prove the effectiveness of modern approaches in the proposed methodology, a set of tasks was tested that contributes to the implementation of interdisciplinary connections and the formation of students' sustainable interest in the study of higher mathematics, digital technologies and programming.

Keywords: geometric fractals, mathematical analysis, affine geometric transformations, programming.

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Introduction

At present, Russian higher education has reached a qualitatively new level of use of digital and information technologies in general and professional education. In this regard, the issues of integration of teaching mathematics and information technology are important in the preparation of future teachers of mathematics and computer science.

Modern pedagogical science pays great attention to integrative processes. The main attention is paid to the issues of the content and technologies of education, the analysis of methodological approaches in the creation and implementation of integrated tasks.

When teaching modern integrated special courses in mathematics, computer support is needed. In this regard, it is important at the present stage of the development of pedagogy and individual methods to focus on such teaching methods and techniques that harmoniously represent the interdisciplinary integration of knowledge.

The course "Geometric fractals" presents the opportunity of using such techniques. The construction of most fractal sets could not be carried out until the second half of the 20th century due to the lack of computer technology and digital technologies. Mathematical knowledge by itself was not enough to construct fractal sets. It was necessary to attract additional knowledge and skills from a related discipline - computer science, since the very nature of fractal sets implies the integration of knowledge in two disciplines.

The connection between mathematics and digital technologies is based on their content. Modern digital technologies make it possible to effectively solve mathematical problems. This is especially true in the process of teaching geometry and mathematical sections related to geometric constructions and their visualization. The ability to visually represent geometric objects is a key feature of computer mathematics systems (CCM), which is most clearly manifested in the process of studying the course "Geometric fractals". On the other hand, there is a universal approach to the construction of various fractals, which is the basis of L-systems that create fractal images. When implementing L-systems, turtle-graphics in Python are used to construct fractal curves on a plane.

Purpose and objectives of the study

In this study, we wanted to explore the features of the course “Geometric Fractals”, which integrates knowledge of higher mathematics and digital technologies, and propose a technology for using affine and projective geometric transformations in constructing fractals using the Maple computer mathematics system, L-systems for fractal curves on the plane using turtle graphics in Python. In this study, we attempted to create individual learning paths and expand the use of digital technologies in teaching the discipline "Geometric Fractals".

Literature review

Fractals are increasingly used in science and practice. The use of fractals in science was reflected in the works of Babkin (2006), Fedoseev (2016), Sekovanov (2013). From a methodological point of view, the application of the theory of fractal sets in education is considered in the publications of Smirnova (2013), Zubova et al. (2008).

The issues of introducing elements of fractal geometry into the practice of school education are described in the works of Dalinger (2001, 2014), Rozov (2005).

The implementation of L-systems for constructing fractal curves is presented, for example, in the works of Feder (2014), Sekovanov et al. (2016), Schroeder (2001), Zubairova et al. (2012).

Fractal figures and their construction in computer mathematics systems on the example of the Matlab package are reflected in the work of Abdumazhitova (2022), Gainutdinova et al. (2017), Gainutdinova and Shirokova (2016a, 2016b), Shirokova (2016), Solonin (2004).

Methodology

When studying fractal geometry, it is impossible to draw a line between mathematics and digital technologies, as it clearly demonstrates their close interweaving, so the Geometric Fractals course integrates students' knowledge of these two disciplines.

The article discusses the methodology associated with the implementation of complex tasks for the construction of geometric fractals, focused on the application of the acquired knowledge in higher mathematics and digital technologies. The complexity of tasks, their multi-stage nature, the possibility of obtaining interesting results and choosing various means and methods for solving a problem should

contribute to the development of students' cognitive research activities. When studying and constructing fractals, students must effectively apply affine geometric transformations, master the basics of programming in high-level languages and computer modeling systems.

In the course of studying the course "Geometric fractals", students should gain knowledge and skills in theory, the alphabet, the laws of constructing L-systems. The L-system is a way of representing recursive structures (such as fractals) as a string of characters and rewriting that string multiple times. With their help, the student can create recursive structures in order to further program them in Python.

A fractal is a geometric figure that has the property of self-similarity, and it consists of parts, each of which is similar to the whole figure as a whole. Geometric fractals are built on the basis of the original figure (line, polygon or polyhedron) by splitting it and performing various transformations of the resulting fragments.

Integrated problems include one of the following geometric fractals: Koch curve, Koch snowflake, square Koch island, crystal, snowflake, Harter-Hateway dragon, Levy curve, Sierpinski triangle, Sierpinski carpet, Sierpinski lattice, tiles, rings, Pythagorean tree, branch, Moore curve, Hilbert curve, Peano curve.

Much attention in complex problems is given to methods for constructing fractal sets. These methods include: construction of fractal objects using affine transformations; generation of fractal objects using L-systems; display of fractal objects on a complex plane. The implementation of these structures is presented using programs computer mathematics system in Maple and turtle graphics in Python.

The study took place in Kazan Federal University in 2020/2021 and 2021/2022 academic years. The course "Geometric fractals" begins with the introduction of the mathematical concept of "fractal object". Its description should be carried out in stages: from the initial idea of a fractal to a mathematical one. First, students become acquainted with fractal objects using the example of the Koch curve, the Koch snowflake, the Levy curve, the Pythagorean tree, etc. In all of the above fractals, students pay special attention to the fact that a small fragment of the boundary of the fractal curve is similar to the original curve. Next, an algorithm for constructing a fractal object is developed in the form of an iterative process. As a result, a fractal is understood as a mathematical object obtained in the course of an iterative process and possessing the property of self-similarity. The concept of a fractal set is considered and refined: a fractal is a self-similar set whose self-similarity dimension is fractional.

The complex task completion process consists of the following steps:

1. Studying the model of this geometric fractal.
2. Derivation of basic formulas using affine geometric transformations.
3. Modeling and construction of fractal objects on a complex plane.
4. Development of a recursive algorithm for constructing a fractal.
5. Implementation of a recursive algorithm for a given geometric fractal in one of the systems of computer mathematics using programming features in a given environment when visualizing a fractal.
6. Application of L-systems to generate self-similar geometric fractals that form systems of repeating functions.
7. Creation of recursive structures with the help of L-systems.
8. Development of programs for recursive structures in Python using the Turtle module.

Consider a classic example of a geometric fractal: the ternary Koch curve (Figure 1).

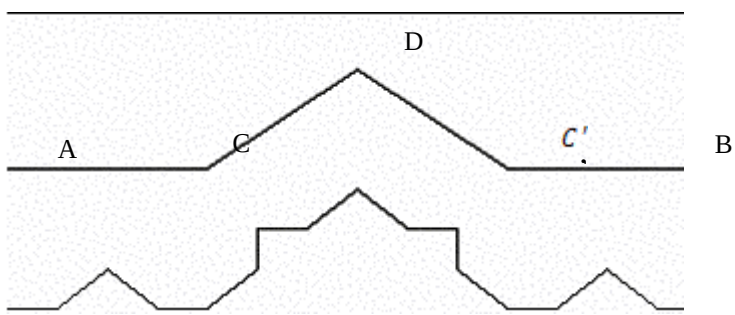

Figure 1. Triadic Koch curve

The Koch curve is a classic geometric fractal, the construction process of which is based on the division of a single segment AB into three equal parts. Next, the middle interval is replaced by an equilateral triangle CDC'. As a result, a broken line is formed from four links of length 1/3. The next stage is a repetition of the previous one for each individual link: a broken line of sixteen links is formed. And so on, resulting in a Koch curve.

Affine transformations are used to obtain the coordinates of points C, D and C'. The coordinates of the points C and C' are obtained using a homothety. These formulas look like this:

$$\begin{cases} X_C = Y_A + \frac{1}{3}(X_B - X_A) \\ Y_C = Y_A + \frac{1}{3}(Y_B - Y_A) \end{cases} \quad (1)$$

$$\begin{cases} X_C' = X_A + \frac{2}{3}(X_B - X_A) \\ Y_C' = Y_A + \frac{2}{3}(Y_B - Y_A) \end{cases} \quad (2)$$

To obtain the coordinates of point D (X', Y'), point A is rotated relative to point C by 120% clockwise using the rotation matrix:

$$\begin{pmatrix} X' \\ Y' \end{pmatrix} = \begin{pmatrix} X_c \\ Y_c \end{pmatrix} + \begin{pmatrix} \cos\frac{2\pi}{3} \cdot \sin\frac{2\pi}{3} \\ -\sin\frac{2\pi}{3} \cdot \cos\frac{2\pi}{3} \end{pmatrix} \cdot \begin{pmatrix} X_A - X_c \\ Y_A - Y_c \end{pmatrix} \quad (3)$$

The result is a formula for point D:

$$\begin{cases} X_C = X_A + \frac{1}{2}(X_B - X_A) - \frac{\sqrt{3}}{6}(Y_B - Y_A) \\ Y_C = Y_A + \frac{1}{2}(Y_B - Y_A) + \frac{\sqrt{3}}{6}(X_B - X_A) \end{cases} \quad (4)$$

According to the obtained formulas (1, 2, 4) for the points C, D and C', an algorithm for constructing a fractal was developed. To construct a fractal set, a cyclic algorithm is needed, which can be described in different ways: both without recursion and with recursion. In this regard, the construction is divided into two stages: an algorithm without recursion (Figure 2) and an algorithm using recursion (Figures 3-5).

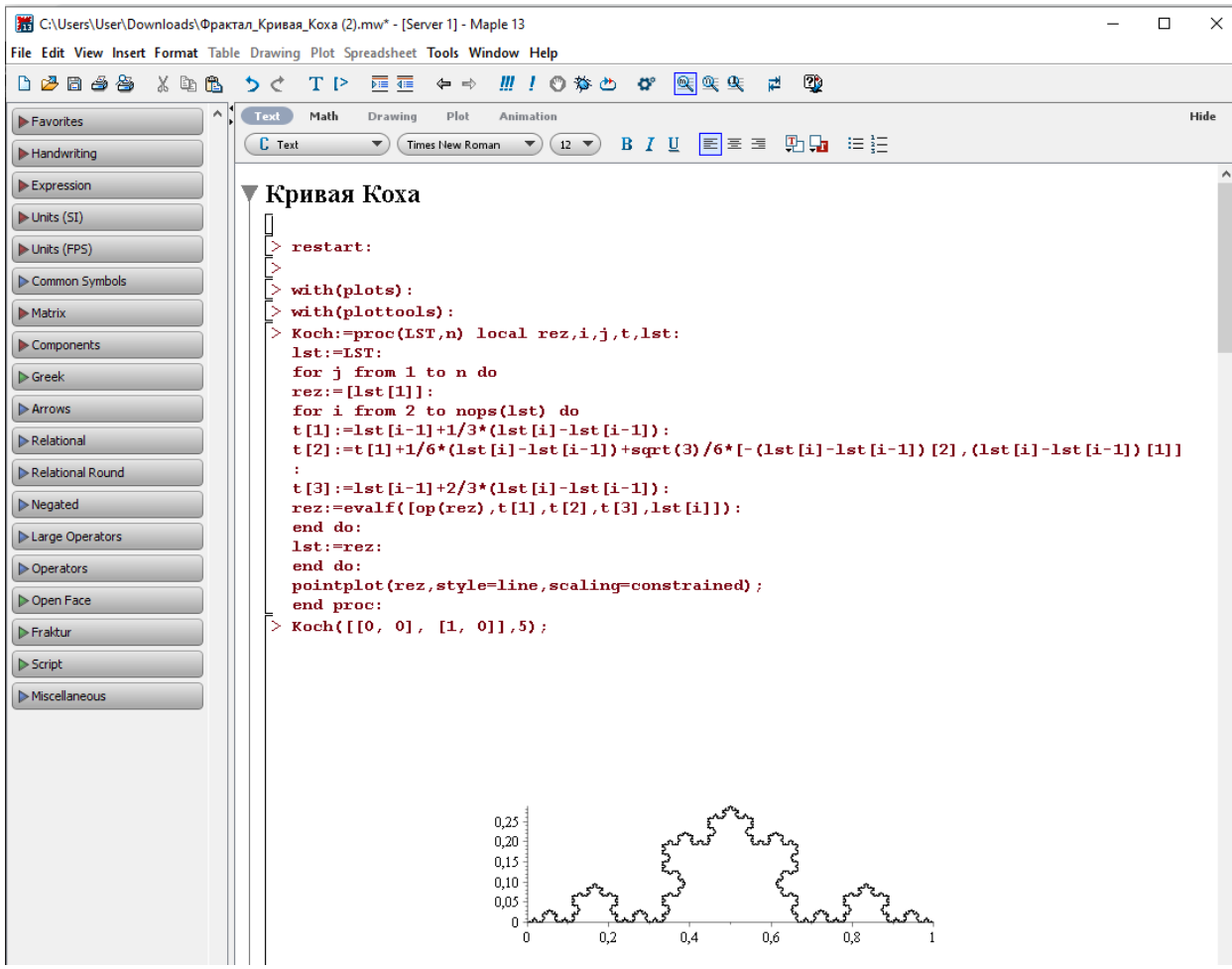

Figure 2. Construction of the Koch curve without recursion

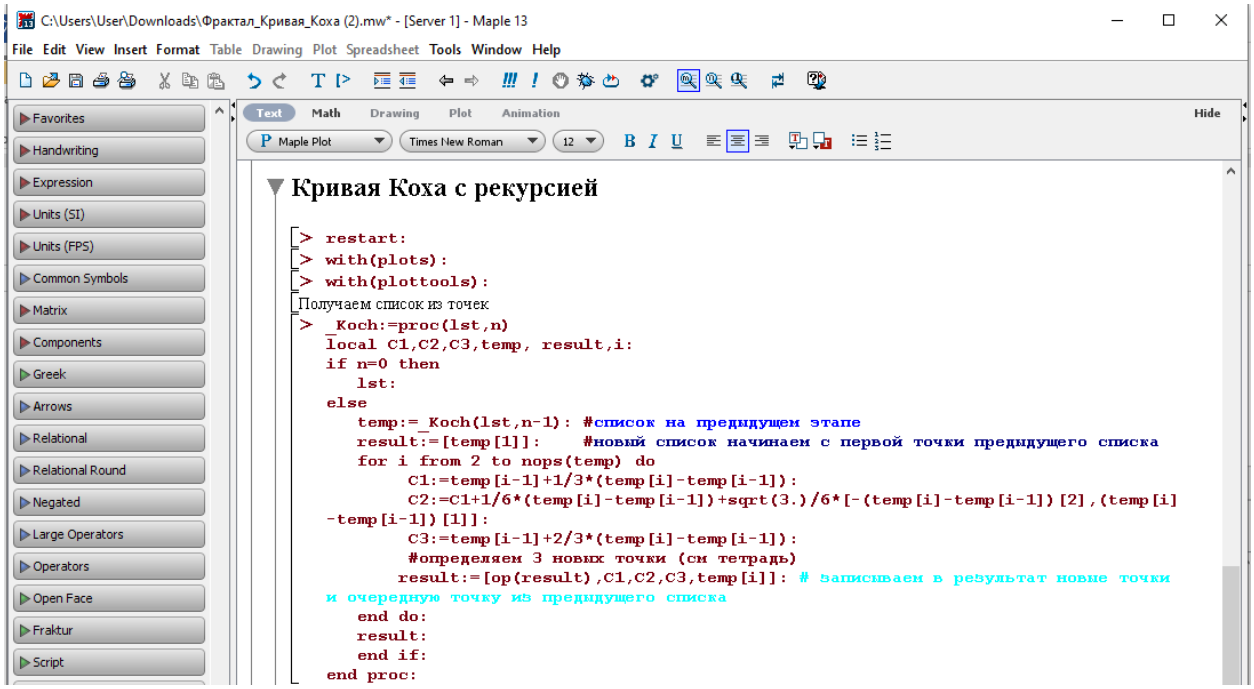

Figure 3. Development of a procedure for constructing a recursive algorithm

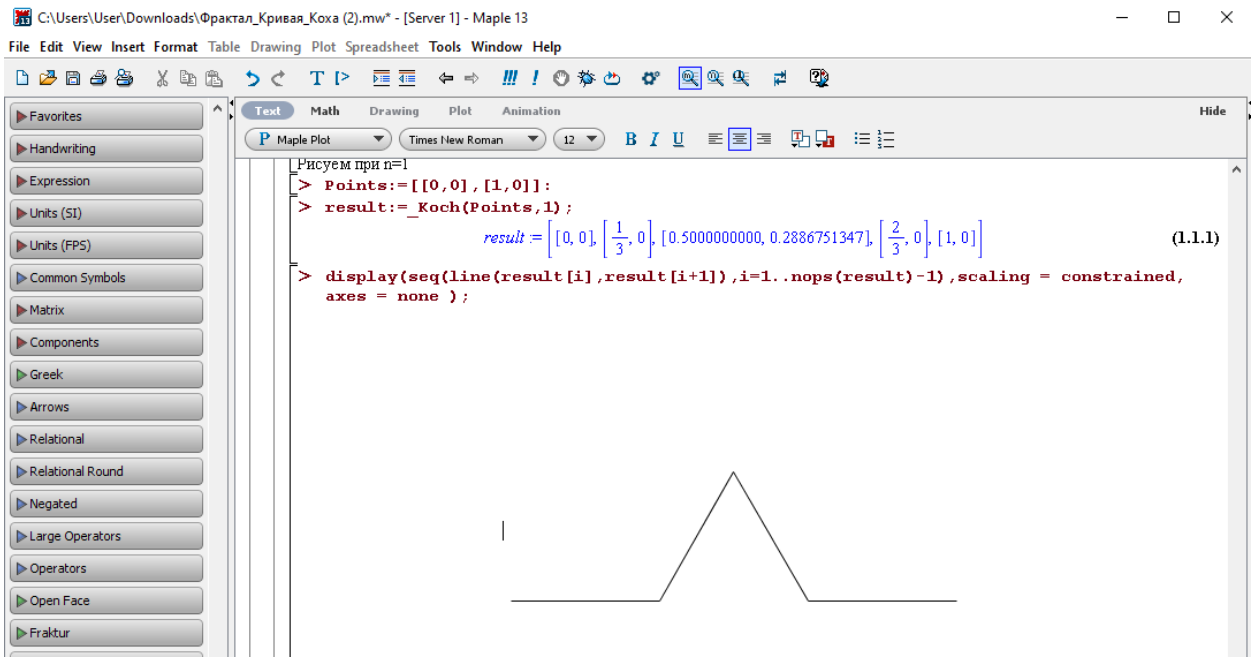

Figure 4. Construction of the Koch curve with recursion (number of iterations n=1)

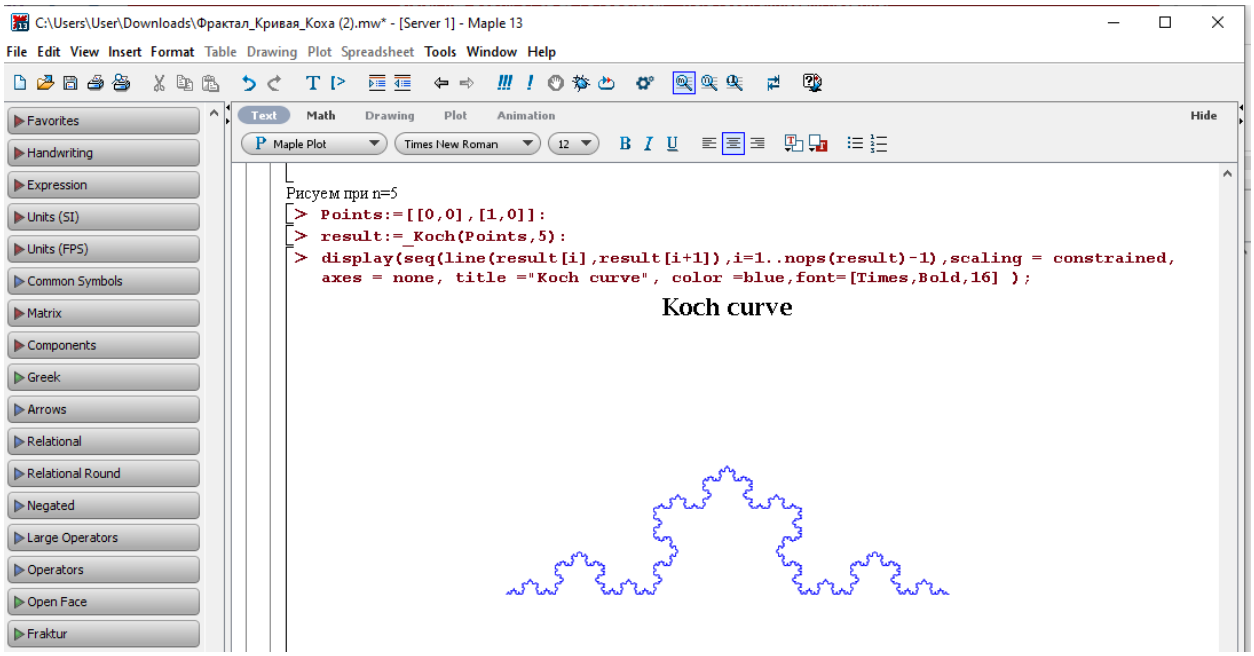

Figure 5. Construction of the Koch curve with recursion (number of iterations n=5)

Having built the Koch curve, it is easy to depict the Koch snowflake, taking as the initial figure not a straight line segment, but an equilateral triangle and dividing into triads on each of its sides (Figure 6).

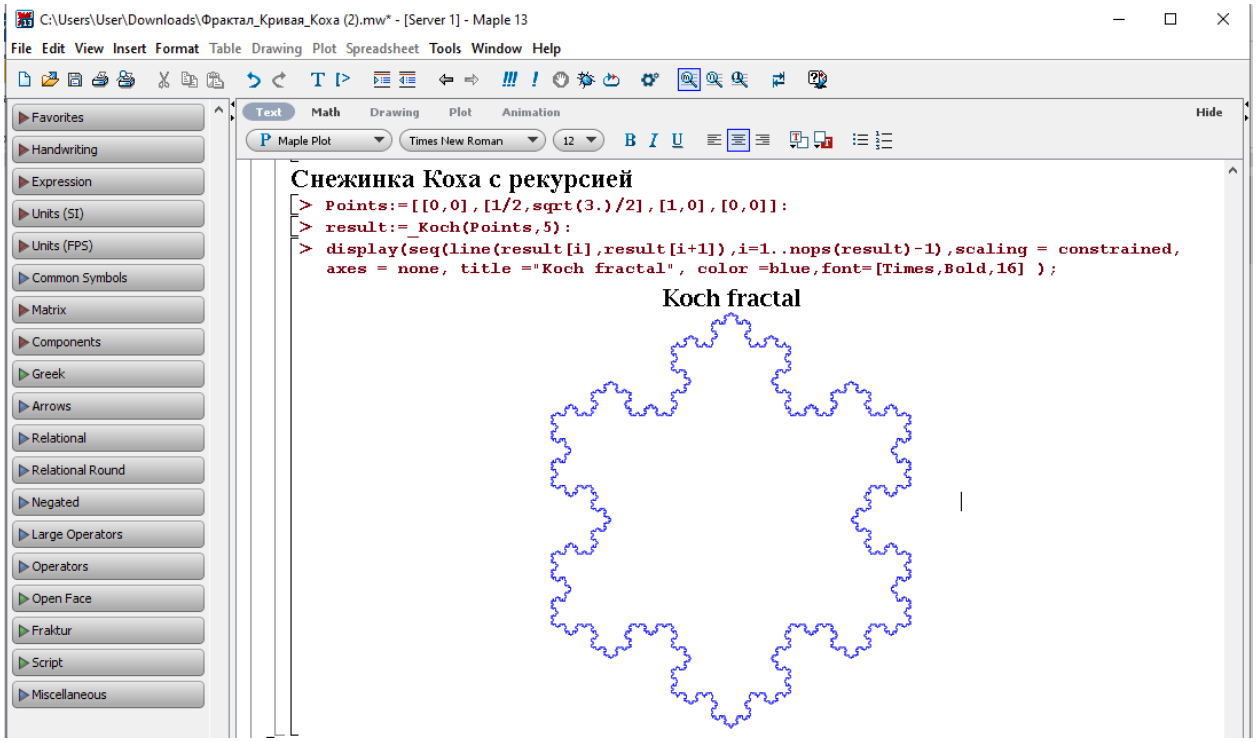

Figure 6. Building a Koch snowflake with recursion (number of iterations n=3)

Further, in the course of studying the course "Geometric fractals", students gain knowledge and skills in theory, alphabet, and the laws of constructing L-systems. The L-system is a way of representing recursive structures (such as fractals) as a string of characters and rewriting that string multiple times.

We can start the construction of the Koch curve from one segment. This is called an axiom and is a 0th order fractal. Then the axiom is replaced according to a certain rule and a fractal of the 1st order is formed. To construct a fractal of the next order, each link is replaced by a generating element. If you start building a curve not from a segment, but from a triangle, and apply the above constructions to each of its sides, you create a Koch snowflake.

Thus, we introduce the notation:

1. "F" – drawing a segment of a given length;

2. "+" – rotation by a given angle clockwise;

3. "-" – rotation by a given angle counterclockwise.

The initial conditions look like:

– axiom: F++F++F;

– rotation angle: Pi/3;

– substitution rule: F-F++F-F.

We are building an L-system. We have F++F++F.

At the first step, replacing all "F" in the axiom with the substitution rule, we see:

F-F++F-F++F-F++F-F++F-F++F-F

On the second step:

F-F++F-F-F-F++F-F++F-F++F-F-F-F++F-F++F-F++F-F-F-F++F-F++F-F++F-F-F-F++F-F++F-F++F-F-F-F++F-F++F-F++F-F-F-F++F-F++F-F++F-F-F-F++F-F

And so on…

After making such substitutions, we get the image of the Koch curve. Thus, the drawing is constructed by recursively applying the inference rules to the axiom (Figure 7).

Let us develop a program for recursive structures in Python using the Turtle module.

```
axiom = "F--F--F"
rules = {"F":"F+F--F+F"}
iterations = 4 # TOP: 7
angle = 60
```

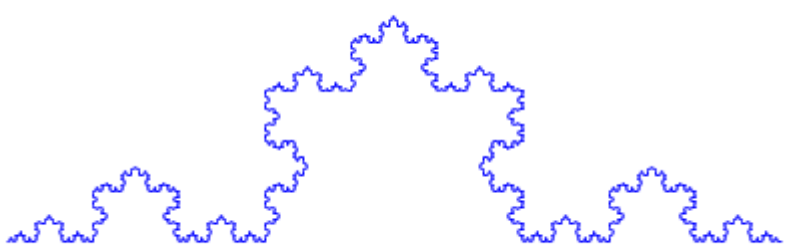

Figure 7. Plotting a Koch Curve with the Turtle Module

Results

This study is aimed at organizing the interaction of a teacher with students, where teaching aids play an important role, allowing: to increase the amount of necessary educational information; improve understanding of the studied material; to systematize knowledge in subjects; develop students' research abilities; create individual trajectories for studying the subject; increase the use of digital technologies in teaching this discipline.

In the course of the study, criteria indicators were measured in 264 students, statistical processing of the results and their interpretation were carried out. Considering the content area of the discipline "Graphic fractals" in this study, 10 basic educational elements are identified: features of the model of a given geometric fractal; formulas for affine geometric transformations; feature of constructing a fractal object on the complex plane; recursive algorithms for constructing a fractal; implementation of the geometric fractal algorithm in computer mathematics systems; programming features; theoretical aspects of L-systems; recursive structures in the L-system; programming in Python using the Turtle module; individual component.

To prove the effectiveness of modern approaches to teaching students according to the proposed methodology, a set of criteria was chosen that reflects the assimilation of theory by students and the development of skills and abilities provided for by the educational standard:

– the excess of the average group indicators obtained as a result of element-by-element and phased analysis of educational material, criterion-based assessment, which has an interdisciplinary character, amounted to $K_\alpha = 0{,}72$ according to the system of complete assimilation of knowledge by Bespalko and Tatur (1989);

– excess of individual indicators obtained as a result of element-by-element and stage-by-stage analysis of educational material on the basis of integrated tasks, the value of the criterion was for the majority of students $K_\alpha = 0{,}7$.

The effectiveness of the results and independence of the student cohort were evidenced by:

– the absence of statistically significant differences in the average group values of the experimental groups of the last two years of study (2020/2021 and 2021/2022 academic years);

– the absence of statistically significant differences in the individual levels of mastering educational material by students of groups who completed their studies in 2021 and 2022.

When analyzing educational material based on integrated tasks, completed by students in the course of studying the discipline "Graphic Fractals" in 2020/2021 and 2021/2022 academic years, the mean of grouped data show that the assimilation of educational material exceeds the criterion value - 70%. The individual indicator of mastering the educational material by each student was checked using element-by-element analysis. The results obtained were correlated with the three identified in accordance with the system of complete assimilation by Bespalko and Tatur (1989); There were no gradations $(I - K_\alpha < 0{,}5; II - 0{,}5 \leq K_\alpha < 0{,}7; III - K_\alpha \geq 0 < 7)$ and the proportion of trainees was more than 70%, which indicates the effectiveness of the proposed methodology.

The stability of the results of applying the proposed methodology and their independence from the group of students was checked by comparing the indicators of elemental analysis for two years of study using Student's t-test. Data analysis allows us to conclude that there is no statistically significant difference in the average group indicators ($T_{\text{эксп.}} = 2{,}086,\ T_{\text{кр.}} = 2{,}134,\ T_{\text{эксп.}} < T_{\text{кр.}}$). This proves the stability of learning when using the proposed methodology.

As a result of the study, the effectiveness of using the developed methodology was proved, which ensures the readiness of future teachers of mathematics and computer science for professional activities.

Discussion

The study considers a methodology related to the implementation of the integrated tasks of the course "Geometric fractals", based on the practical application of the knowledge gained by students in higher mathematics and digital technologies.

The proposed methodological approach defines the following tasks:

1. Taking into account the analysis of scientific and methodological sources, a methodological approach to a large fractal set is determined based on the use of mathematics and digital technologies in teaching.

2. The developed complex tasks are aimed at improving the quality of knowledge in mathematics, digital technologies and programming through the study of geometric fractals.

3. The presented methodology contributes to the implementation of interdisciplinary connections and forms a steady interest in the study of mathematics.

4. The course "Geometric fractals" contributes to the development of students' research competencies, as well as increases the level of their logical thinking.

5. This technique expands students' knowledge by including new concepts in the course: fractal sets, L-systems, etc., and also forms practical skills in constructing fractal curves on a plane using programs in the Maple computer mathematics system and turtle graphics in the Python language.

The use of research activities and complex tasks in modern education is determined by their multifunctional orientation, as well as the possibility of integration into a holistic educational process.

The proposed course "Geometric fractals" with integrated tasks contributes to the development of skills to apply digital technologies in the construction of mathematical objects.

The introduction of this technique is recommended in connection with the current level of development of digital technologies and mathematics. The creation and development of methods based on the systematic use of digital technologies is a necessary and timely step towards increasing the effectiveness of mathematical education.

Conclusion

Modern educational technologies, which include new innovative forms, allow you to move to a higher quality level of education. The presented methodology contributes to the implementation of interdisciplinary connections and forms a steady interest of students in the study of higher mathematics, digital technologies and programming.

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